Chapter 14
The Tanks-In-Series Model

- We have learnt how to calculate the conversion in a series of mixed flow reactors. And we know when the number of tanks go to infinity, we get a plug flow performance.
- Here we have a non-ideal flow reactor, and suppose there were some equal-size mixed flow reactors, they have same performance as the non-ideal reactor.
- We try to find how many they are in this chapter.

- 14.1 Pulse Response Experiments and the RTD

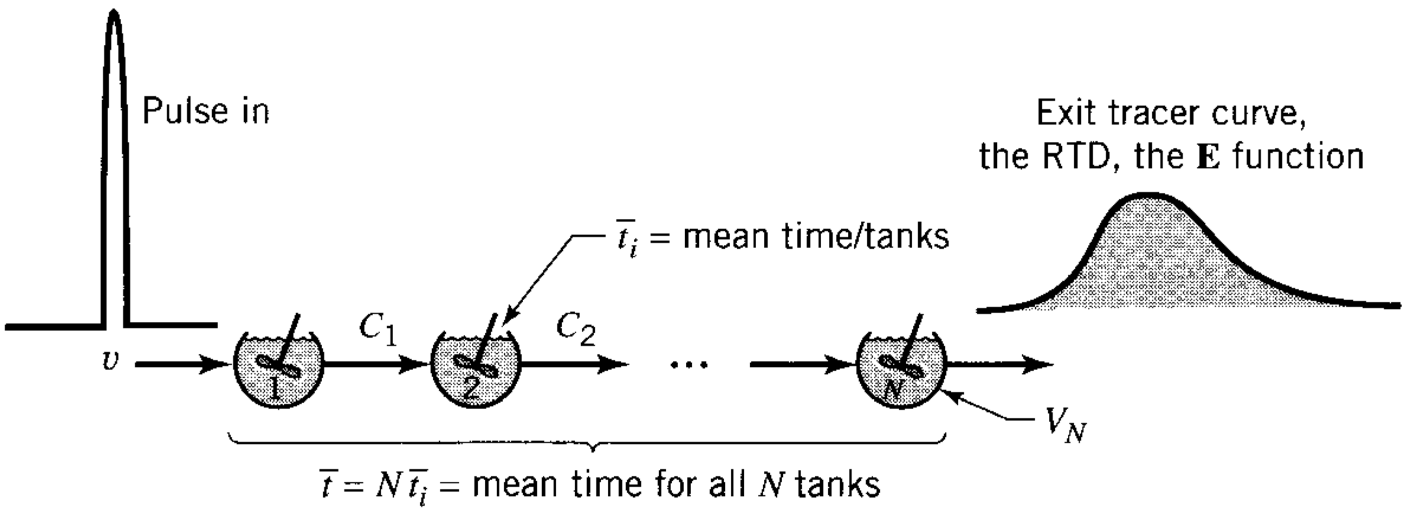


Figure 14.1 The tanks-in-series model.

- For the first tank at any time after t

$$\begin{pmatrix}\text{rate of disappearance} \\ \text{of tracer}\end{pmatrix} = \begin{pmatrix}\text{input} \\ \text{rate}\end{pmatrix} - \begin{pmatrix}\text{output} \\ \text{rate}\end{pmatrix}$$

$$\begin{pmatrix}\text{rate of disappearance} \\ \text{of tracer}\end{pmatrix} = \begin{pmatrix}\text{input} \\ \text{rate}\end{pmatrix} - \begin{pmatrix}\text{output} \\ \text{rate}\end{pmatrix}$$

$$V_1 \frac{dC_1}{dt} = 0 - vC_1 \qquad [mol\ tracer/s]$$

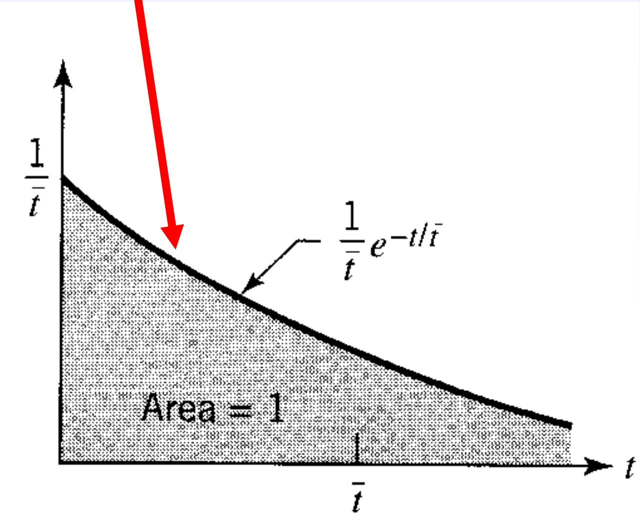
Important: This is not a steady-state

$$\int_{C_0}^{C_1} \frac{dC_1}{C_1} = -\frac{1}{\bar{t}_1}\int_0^t dt \quad \Rightarrow \quad \frac{C_1}{C_0} = e^{-\frac{t}{\bar{t}_1}}$$

$$\int_0^\infty \frac{C_1}{C_0} dt = \int_0^\infty e^{-\frac{t}{\bar{t}}} dt = \bar{t}$$

$$\bar{t}_1 E_1 = e^{-\frac{t}{\bar{t}_1}} \qquad [N = 1]$$

See Fig. 11.14



For the second tank

$$V_2 \frac{dC_2}{dt} = vC_1 - vC_2 \qquad [mol\ tracer/s]$$

$$V_2 \frac{dC_2}{dt} = vC_0 e^{-\frac{t}{\bar{t}_1}} - vC_2$$

$$\bar{t}_2 E_2 = \frac{t}{\bar{t}_2} e^{-\frac{t}{\bar{t}_2}} \qquad [N = 2]$$

For the Nth tank

$$\bar{t} E = \left(\frac{t}{\bar{t}}\right)^{N-1} \frac{N^N}{(N-1)!} e^{-\frac{tN}{\bar{t}}} \qquad \bar{t} = N\bar{t}_i \qquad \sigma^2 = \frac{\bar{t}^2}{N}$$

For the Nth tank

$$\bar{t}E = \left(\frac{t}{\bar{t}}\right)^{N-1} \frac{N^N}{(N-1)!} e^{-\frac{tN}{\bar{t}}} \qquad \bar{t} = N\bar{t}_i \qquad \sigma^2 = \frac{\bar{t}^2}{N}$$

$$\bar{t}_i E = \left(\frac{t}{\bar{t}_i}\right)^{N-1} \frac{1}{(N-1)!} e^{-\frac{t}{\bar{t}_i}} \qquad \overline{t_i} = \frac{\bar{t}}{N} \qquad \sigma^2 = N\bar{t}_i^2$$

$$E_{\theta i} = \bar{t}_i E = \frac{\theta_i^{N-1}}{(N-1)!} e^{-\theta_i} \qquad \sigma_{\theta i}^2 = N \qquad \left(\theta_i = \frac{t}{\bar{t}_i}\right)$$

$$E_\theta = (N\bar{t}_i)E = N\frac{(N\theta)^{N-1}}{(N-1)!} e^{-N\theta} \qquad \sigma_\theta^2 = \frac{1}{N} \qquad \left(\theta = \frac{t}{\bar{t}}\right)$$

For N tanks

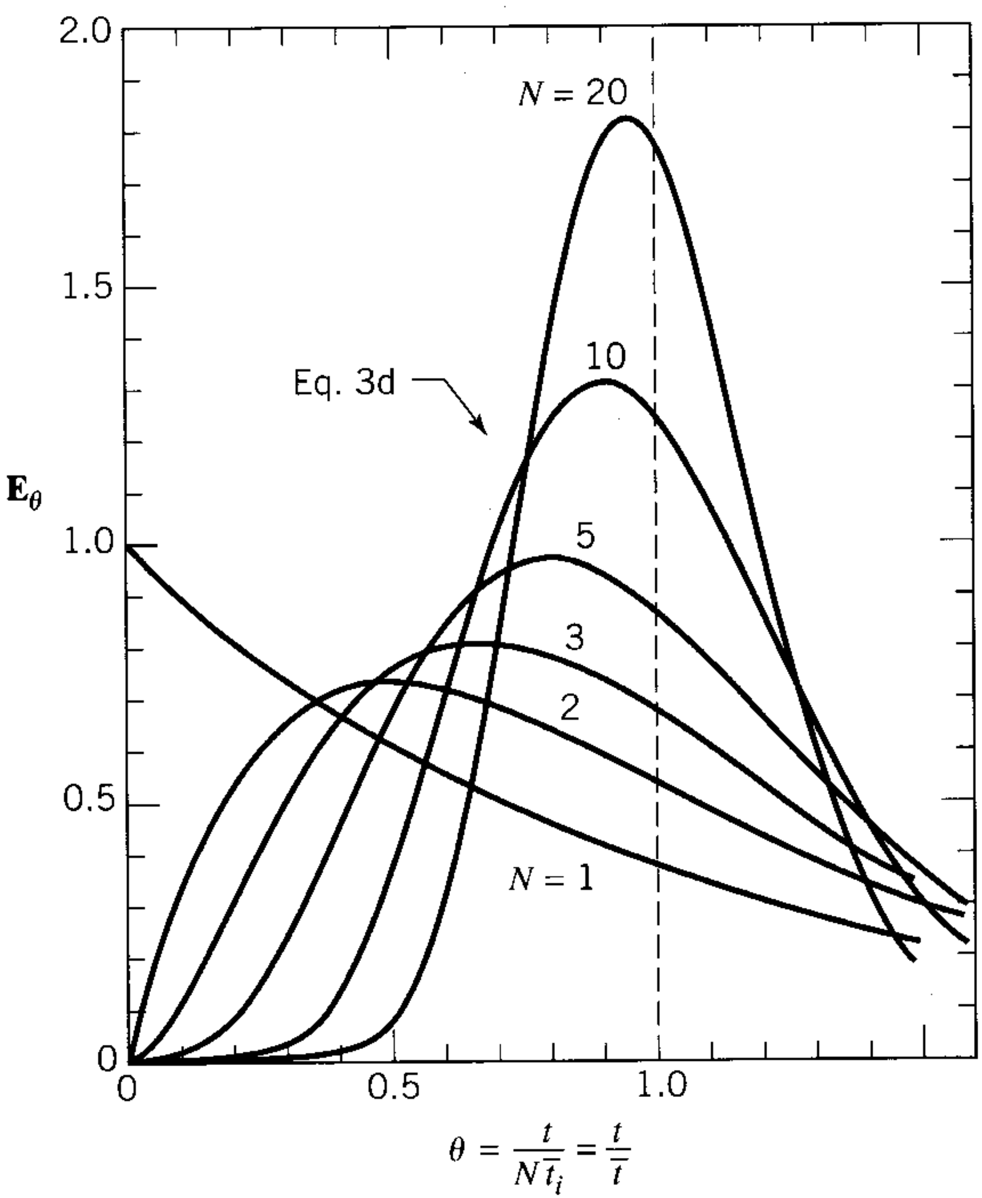
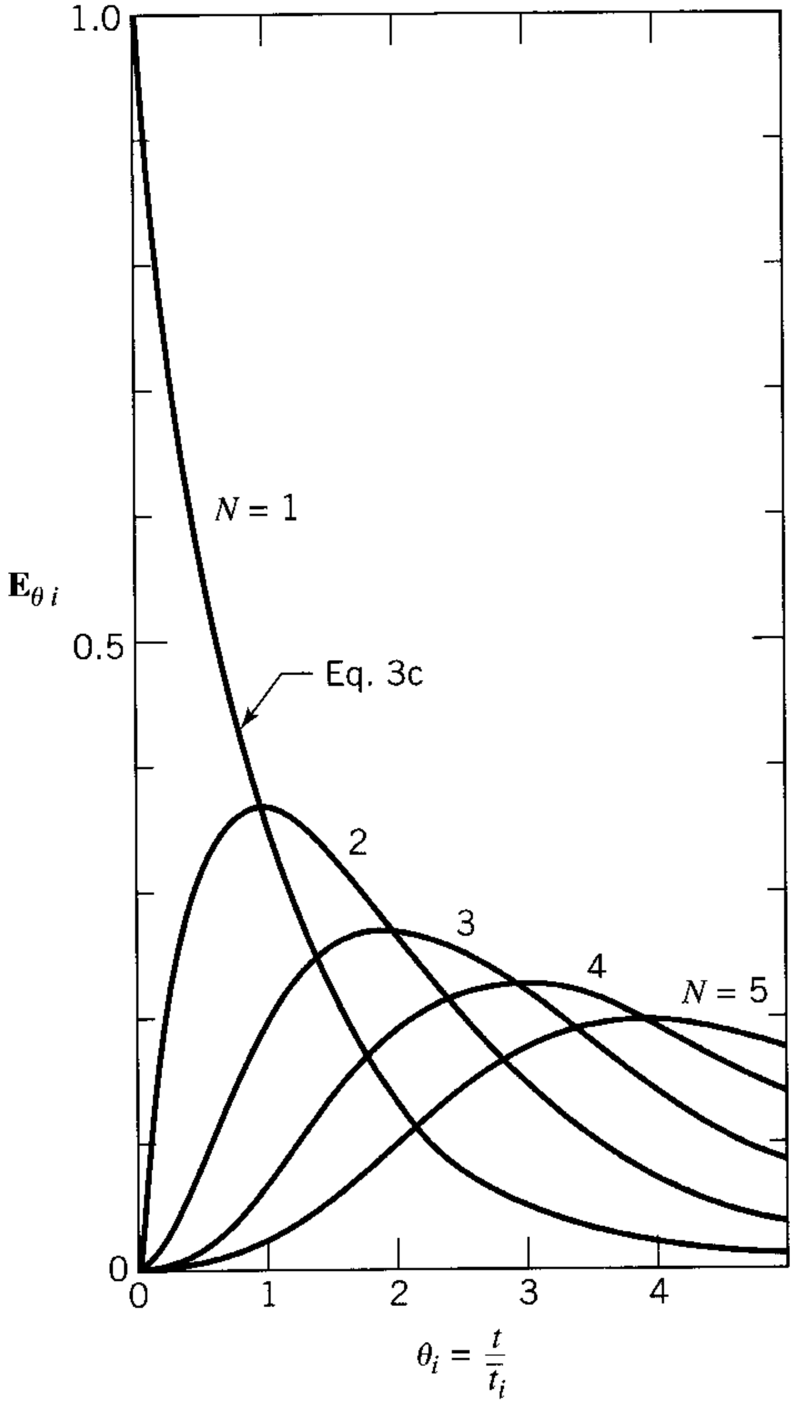


Figure 14.2 RTD curves for the tanks-in-series model, Eq. 3.



For the Nth tank

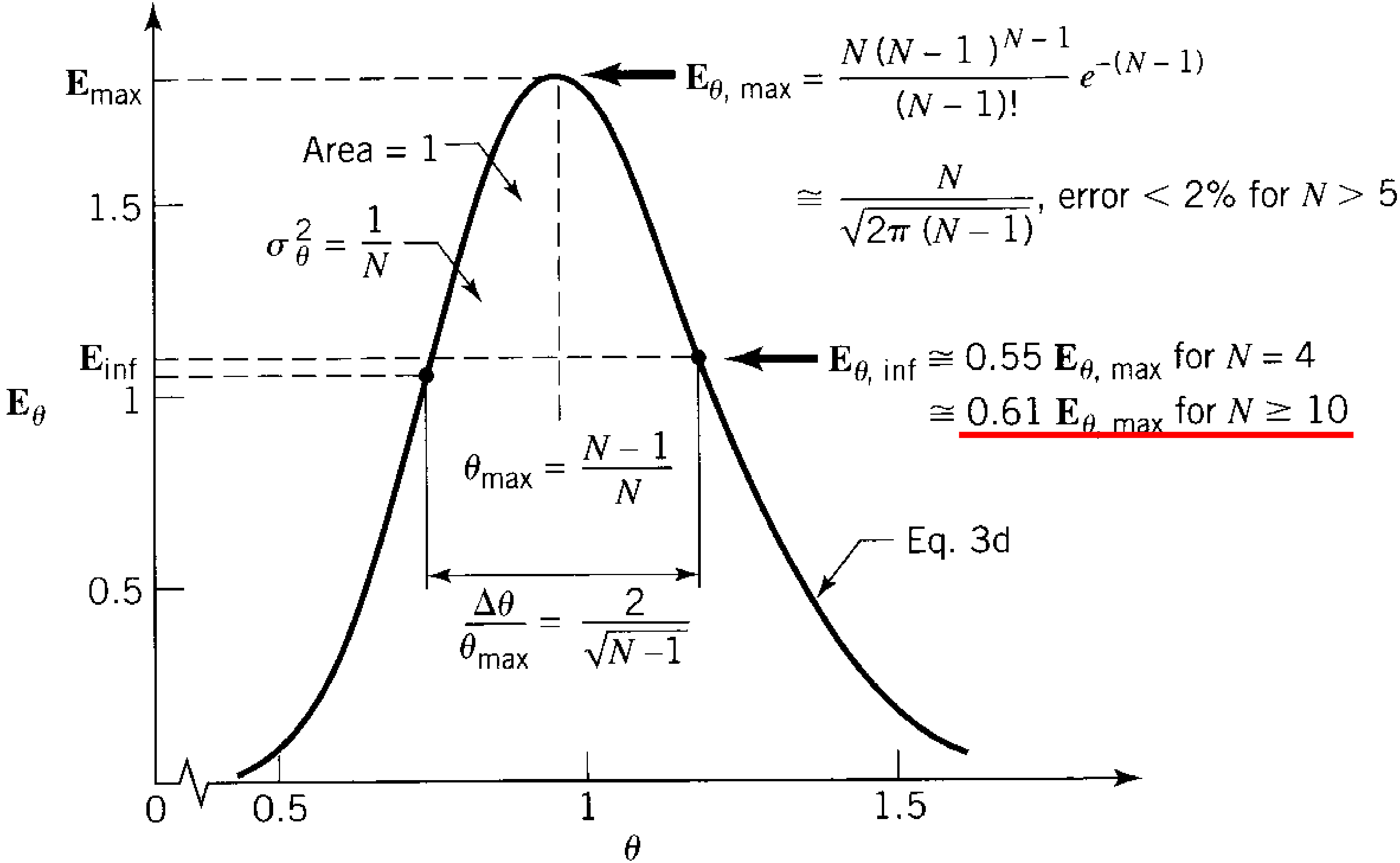


Figure 14.3 Properties of the RTD curve for the tanks-in-series model.

- Comparison between dispersion and tank-in-series models
- The link of two models

$$\sigma_\theta^2 = \frac{1}{N}$$ tank in series model

$$\sigma_\theta^2 = 2\frac{D}{uL}$$ dispersion model with small diviation from plug flow

$$\sigma_\theta^2 = 2\left(\frac{D}{uL}\right) - 2\left(\frac{D}{uL}\right)^2 \left(1 - \exp\left(-uL / D\right)\right)$$ dispersion model with closed boundary condition

$$\sigma_\theta^2 = 2\frac{D}{uL} + 8\left(\frac{D}{uL}\right)^2$$ dispersion model with open-open condition

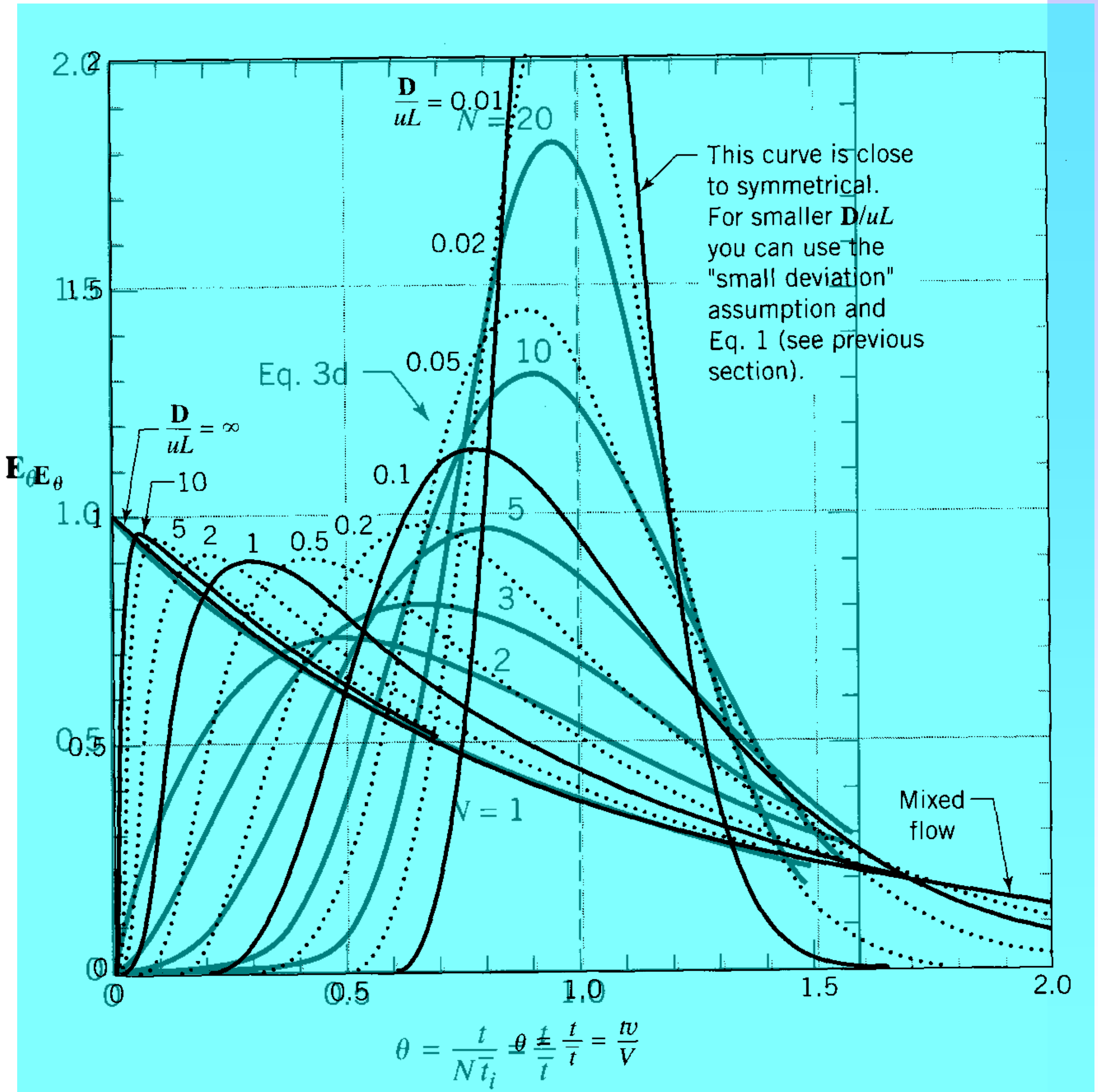


Figure 14.2 RTD curves for the tanks-in-series model, Eq. 3.

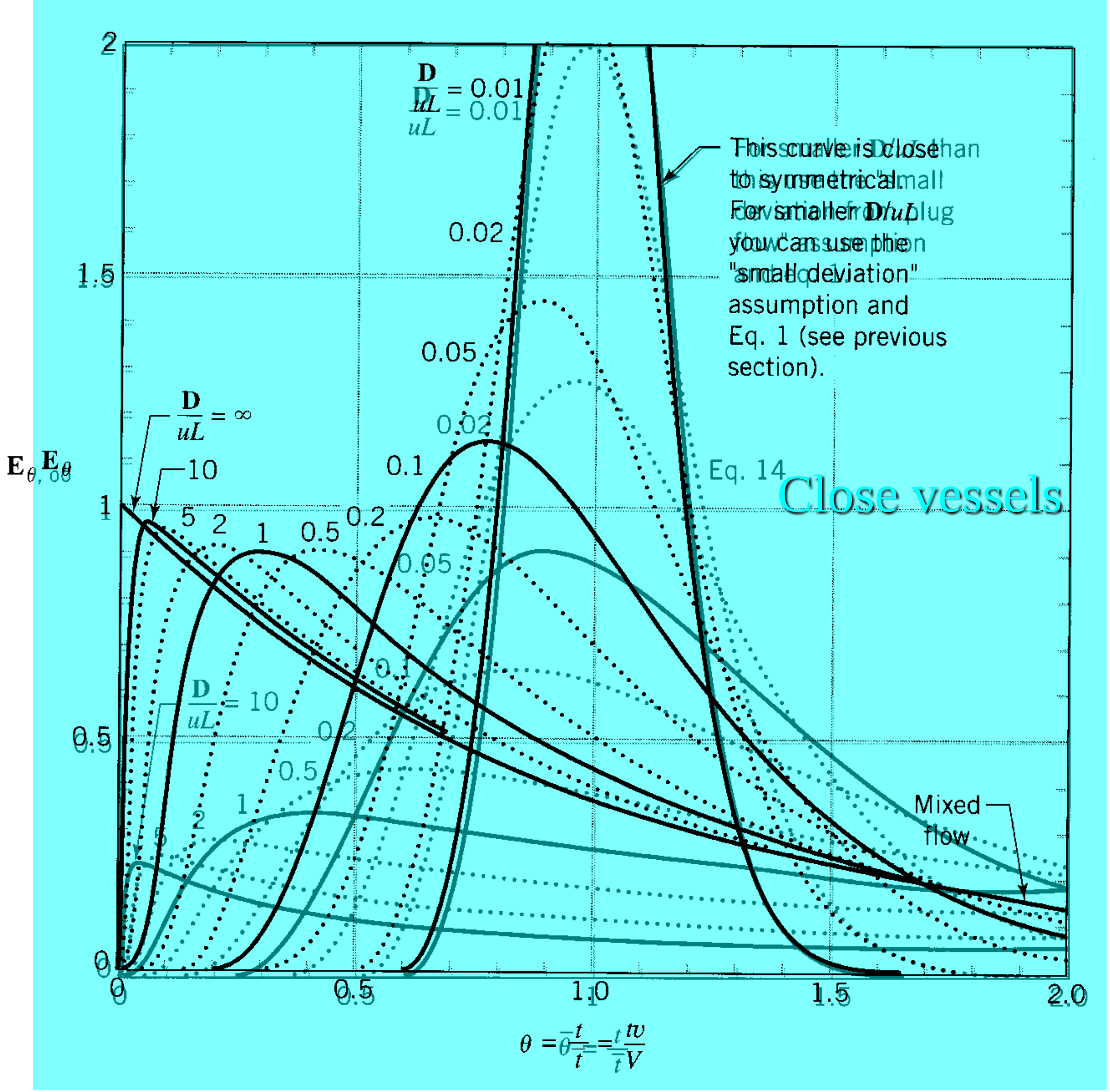


Figure 13.10 Tracer response curves for "open" vessels having large deviations from plug flow.

- Comments and Extensions
- Independence
- Independence means that the fluid loses its memory as it passes from vessel to vessel. Thus a faster moving fluid element in one vessel does not remember this fact in the vessel and doesn’t preferentially flow faster there. Laminar flow often does not satisfy this requirement; however, complete mixing of fluid between units satisfies this condition.

- With the per-condition of independence we have
- if M tanks are connected to N more tanks (all of the same size) then the individual means and variance (in ordinary time units) are additive

$$\bar{t}_{M+N} = \bar{t}_M + \bar{t}_N \qquad and \qquad \sigma^2_{M+N} = \sigma^2_M + \sigma^2_N$$

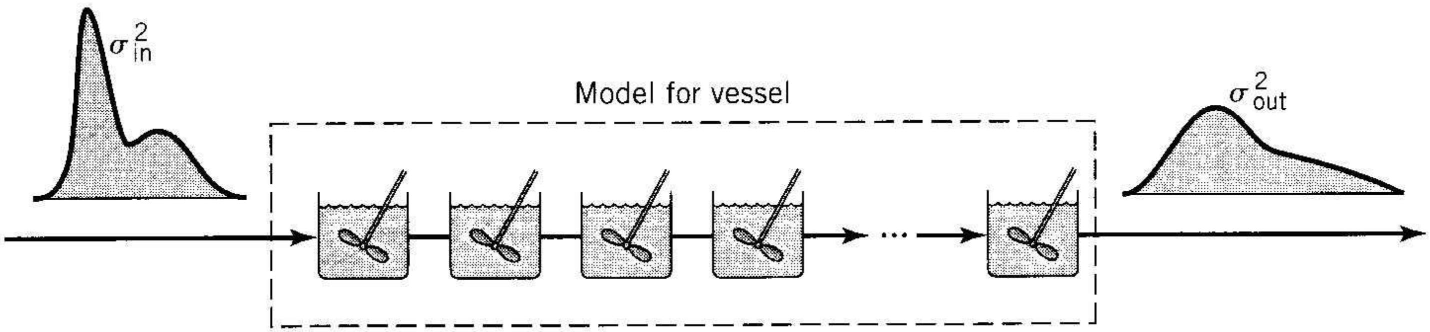
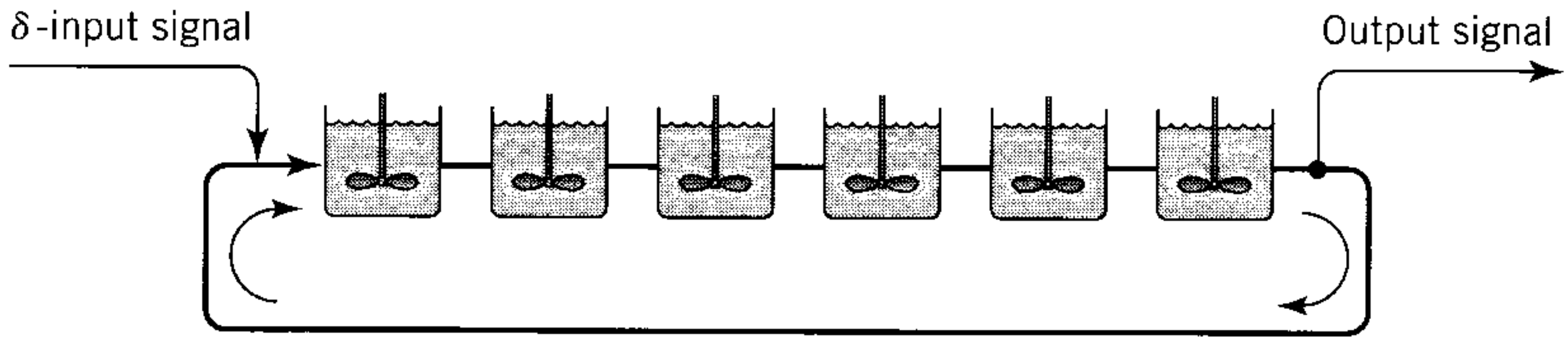


Figure 14.4 For any one-shot tracer input Eq. 4 relates input, output, and number of tanks.

- If we introduce any one-shot tracer input into N tanks, we can write

$$\Delta\sigma^2 = \sigma^2_{out} - \sigma^2_{in} = \frac{(\Delta\bar{t})^2}{N}$$

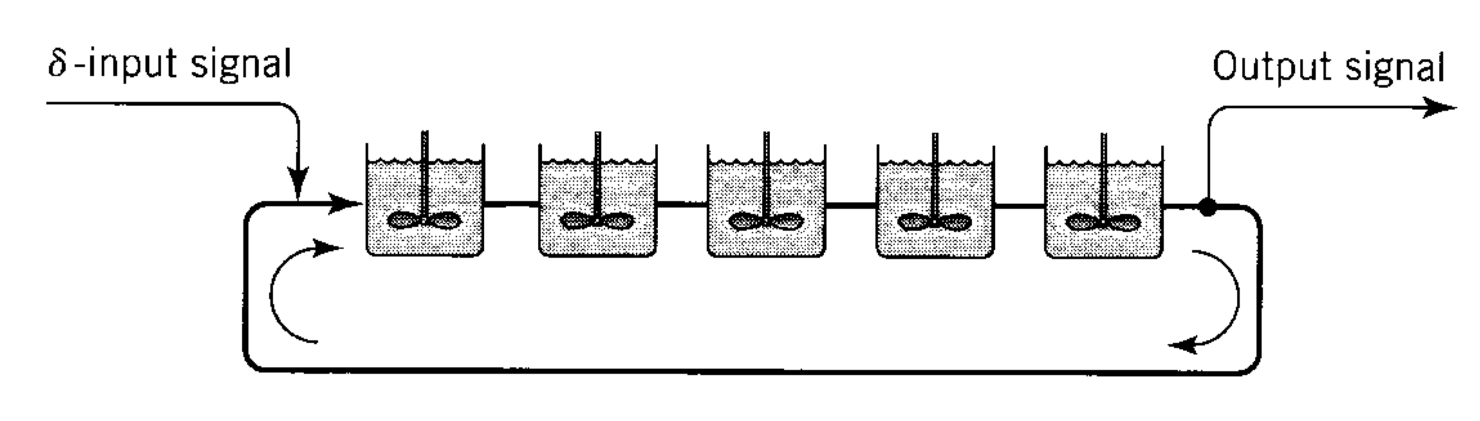
- Close Recirculation System



$$\bar{t}_i C_{\text{pulse}} = e^{-t/\bar{t}_i} \sum_{m=1}^{\infty} \frac{(t/\bar{t}_i)^{mN-1}}{(mN-1)!}$$

$$C_{\theta i,\,\text{pulse}} = e^{-\theta_i} \sum_{m=1}^{\infty} \frac{\theta_i^{mN-1}}{(mN-1)!}$$

$$C_{\theta,\,\text{pulse}} = N e^{-N\theta} \sum_{m=1}^{\infty} \frac{(N\theta)^{mN-1}}{(mN-1)!}$$



- If we have 5 tanks in series

$$C_{\text{pulse}} = \frac{5}{\bar{t}} e^{-5t/\bar{t}_i} \left[\frac{(5t/\bar{t})^4}{4!} + \frac{(5t/\bar{t})^9}{9!} + \cdots \right]$$

$$C_{\theta i,\,\text{pulse}} = e^{-\theta_i} \left[\frac{\theta_i^4}{4!} + \frac{\theta_i^9}{9!} + \frac{\theta_i^{14}}{14!} + \cdots \right]$$

$$C_{\theta,\,\text{pulse}} = 5e^{-5\theta} \left[\frac{(5\theta)^4}{4!} + \frac{(5\theta)^9}{9!} + \cdots \right]$$

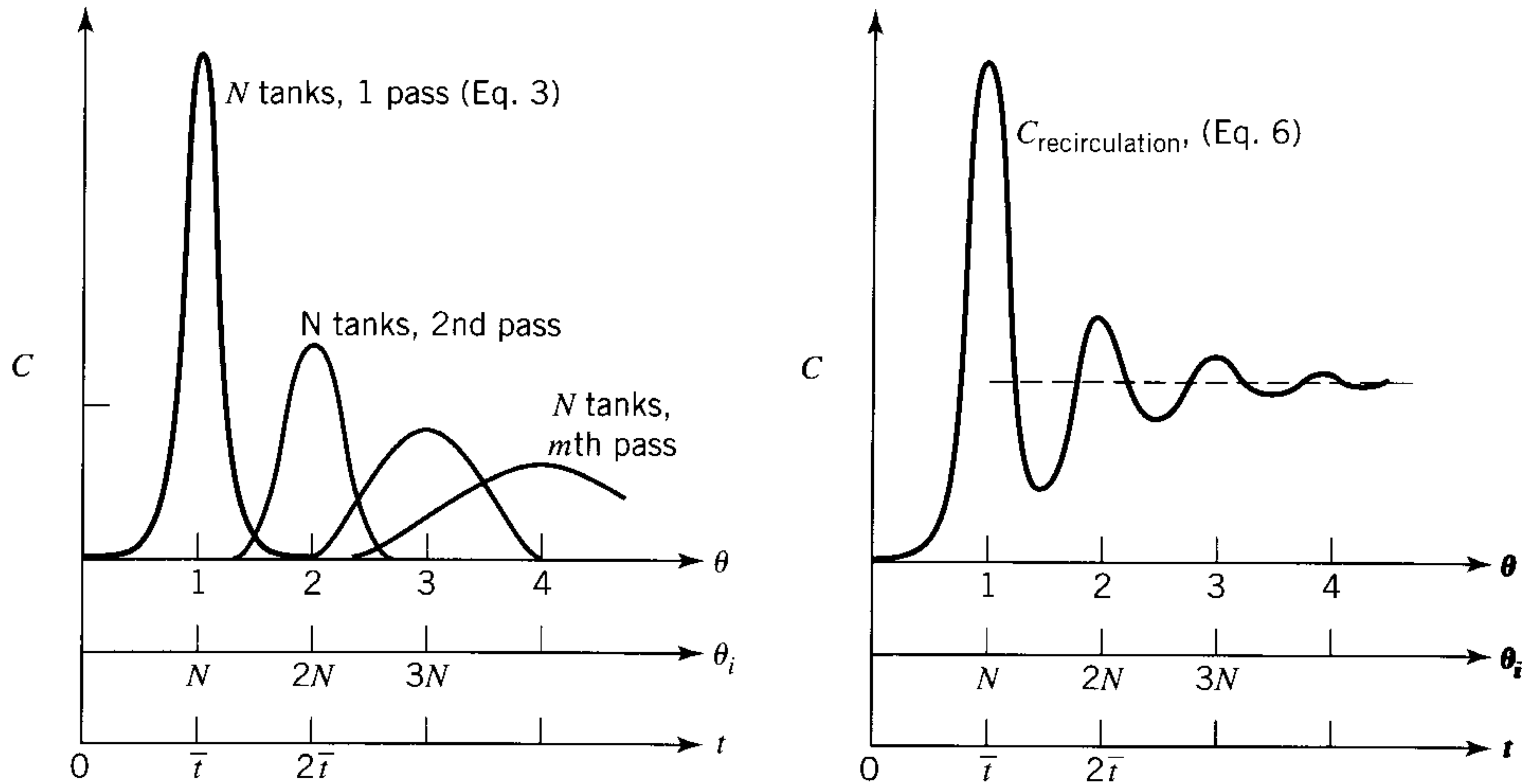


Figure 14.5 Tracer signal in a recirculating system.

• Recirculation with Throughflow

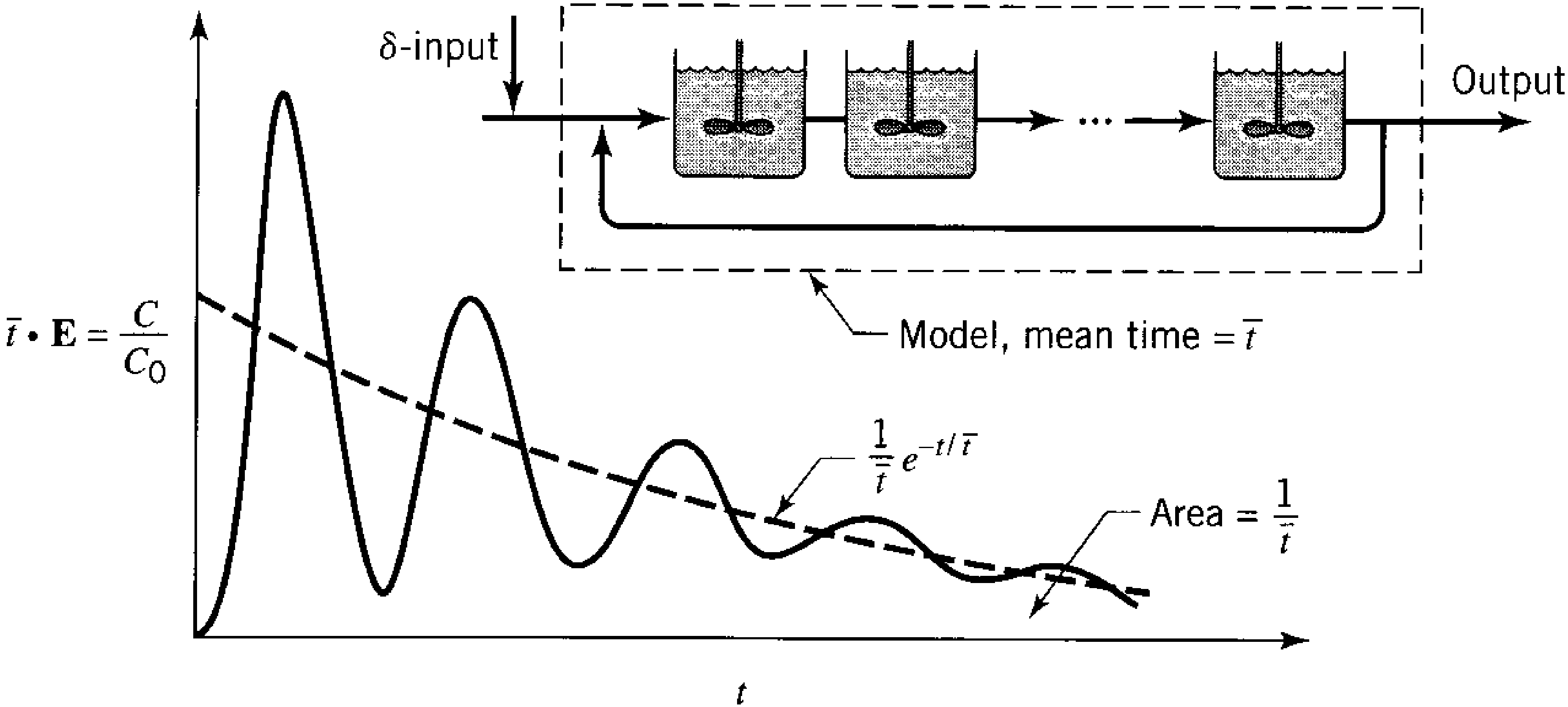
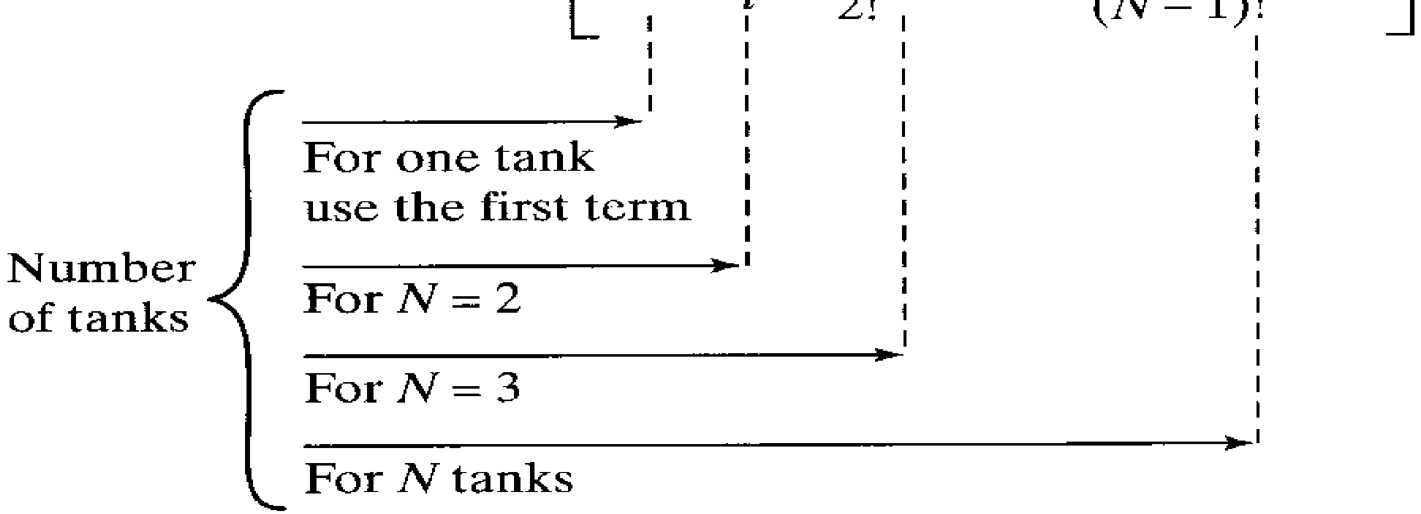


Figure 14.6 Recirculation with slow throughflow.

- Step Response Experiments and the F Curve
- The output F curve from a series of N ideal stirred tanks is

$$\mathbf{F} = 1 - e^{-N\theta}\left[1 + N\theta + \frac{(N\theta)^2}{2!} + \cdots + \frac{(N\theta)^{N-1}}{(N-1)!} + \cdots\right]$$

$$\mathbf{F} = 1 - e^{-\theta_i}\left[1 + \theta_i + \frac{\theta_i^2}{2!} + \cdots + \frac{\theta_i^{N-1}}{(N-1)!} + \cdots\right]$$



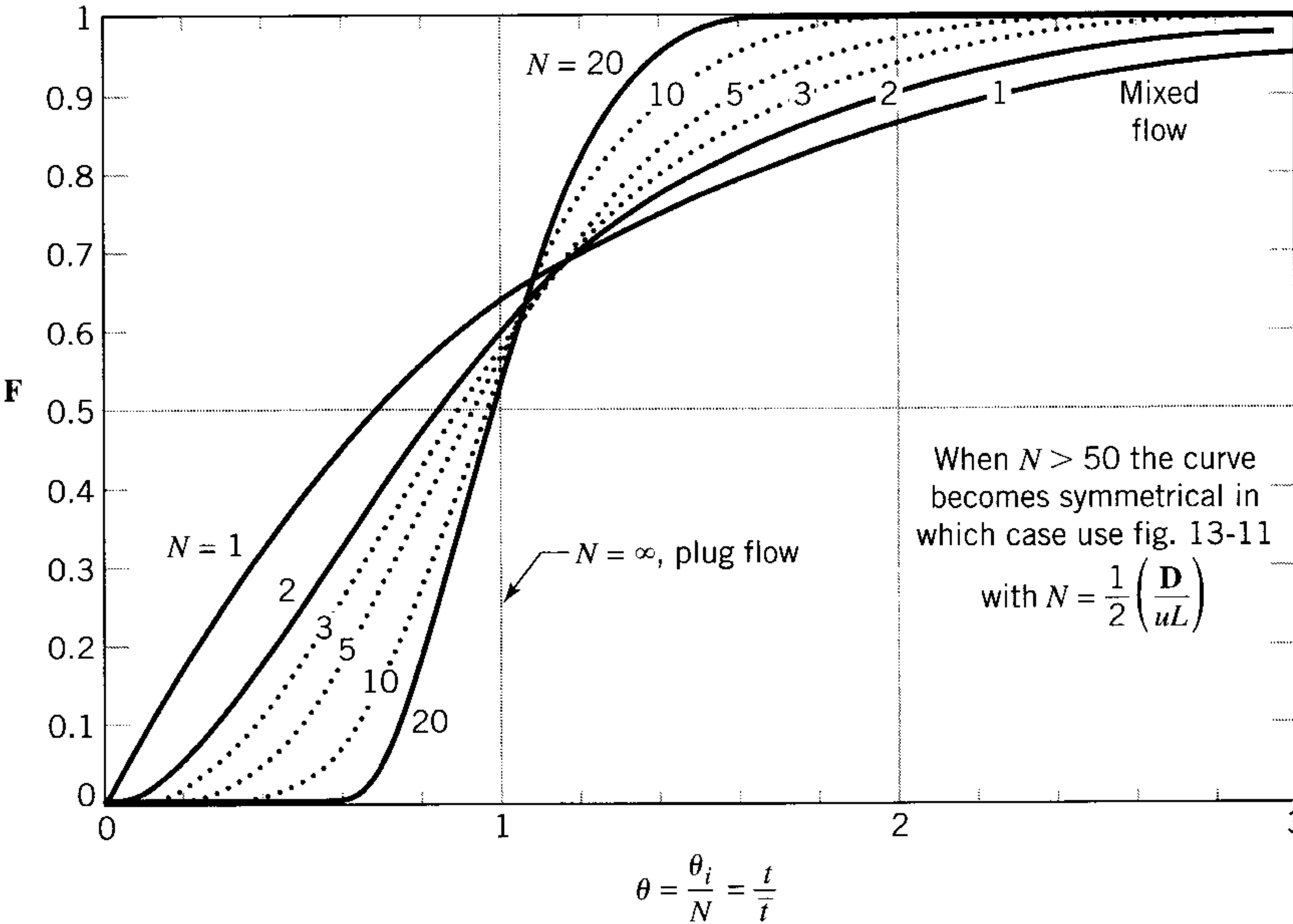


Figure 14.7 The **F** curve for the tanks-in-series model, from MacMullin and Weber (1935).

- 14.2 Chemical Conversion
- The conversion equation has been developed in chapter 6
- For a single tank:
- First order reaction

$$\frac{C_A}{C_{A0}} = \frac{1}{1+k\bar{t}_i} = \frac{1}{1+k\bar{t}}$$

- For N tanks in series:

$$\frac{C_A}{C_{A0}} = \frac{1}{\left(1+k\bar{t}_i\right)^N} = \frac{1}{\left(1+\frac{k\bar{t}}{N}\right)^N}$$

For small deviations from plug flow (large N)

comparision with plug flow

For same $C_{A\ final}$

$$\frac{V_{N\ tanks}}{V_P} = 1 + k\bar{t}_i = 1 + \frac{k\bar{t}}{2N}$$

For same volume V

$$\frac{C_{N\ tanks}}{C_{AP}} = 1 + \frac{(k\bar{t})^2}{2N}$$

$$\frac{C_A}{C_{A0}} = \frac{1}{(1+k\bar{t}_i)^N} = \frac{1}{\left(1+\frac{k\bar{t}}{N}\right)^N} \quad \text{For N tanks}$$

$$\frac{C_A}{C_{A0}} = \exp(-k\tau_P) \quad \text{For plug flow}$$

$$-k\tau_P = \ln\frac{C_A}{C_{A0}} = -N\ln\left(1+\frac{k\bar{t}}{N}\right) \qquad \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$$

$$-k\tau_P = \ln\frac{C_A}{C_{A0}} = -N\left(\frac{k\bar{t}}{N} - \frac{1}{2}\left(\frac{k\bar{t}}{N}\right)^2\right) \quad \text{taking two iterms only}$$

$$\frac{\bar{t}}{\tau_P} = \frac{1}{1-\frac{1}{2}\frac{k\bar{t}}{N}} \qquad \frac{1}{1-x} = 1 + x + x^2 + \ldots$$

$$\frac{\bar{t}}{\tau_P} = 1 + \frac{k\bar{t}}{2N} = \frac{V_{\text{N tanks}}}{V_p}$$

For small deviations from plug flow (large N)

comparision with plug flow

For same volume V $\quad \dfrac{C_{N\text{ tanks}}}{C_{AP}} = 1 + \dfrac{(k\bar{t})^2}{2N} \qquad \left(\dfrac{C_A}{C_{A0}}\right)_P = e^{-k\tau}$

$$\left(\frac{C_A}{C_{A0}}\right)_N = \frac{1}{\left(1+\frac{k\bar{t}}{N}\right)^N} \qquad \lim_{N\to\infty}\frac{1}{\left(1+\frac{k\bar{t}}{N}\right)^N} \qquad \text{let } x = \frac{N}{k\tau}$$

$$\lim_{N\to\infty}\frac{1}{\left(1+\frac{k\bar{t}}{N}\right)^N} = \lim_{x\to\infty}\frac{1}{\left(1+\frac{1}{x}\right)^{xk\tau}} = \lim_{x\to\infty}\left[\left(1+\frac{1}{x}\right)^x\right]^{-k\tau}$$

since $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$ therefore $\lim_{N\to\infty}\dfrac{1}{\left(1+\frac{k\bar{t}}{N}\right)^N} = e^{-k\tau}$

- Second-Order Reaction of a Microfluid

For $A \rightarrow R$ or $A + B \rightarrow R$ with $C_{A0} = C_{B0}$

We have learnt in Chapter 6 already

$$C_N = \frac{1}{4k\tau_i}\left(-2 + 2\sqrt{\vdots \atop -1 \cdots + 2\sqrt{-1 + 2\sqrt{1 + 4kC_0 k\tau_i}}} \right\}N \Bigg)$$

For all other reaction kinetics of microfluids,

$\bar{t}_i = \dfrac{C_{Ai-1} - C_{Ai}}{-r_i}$ have to be solved numerically

or graphcally.

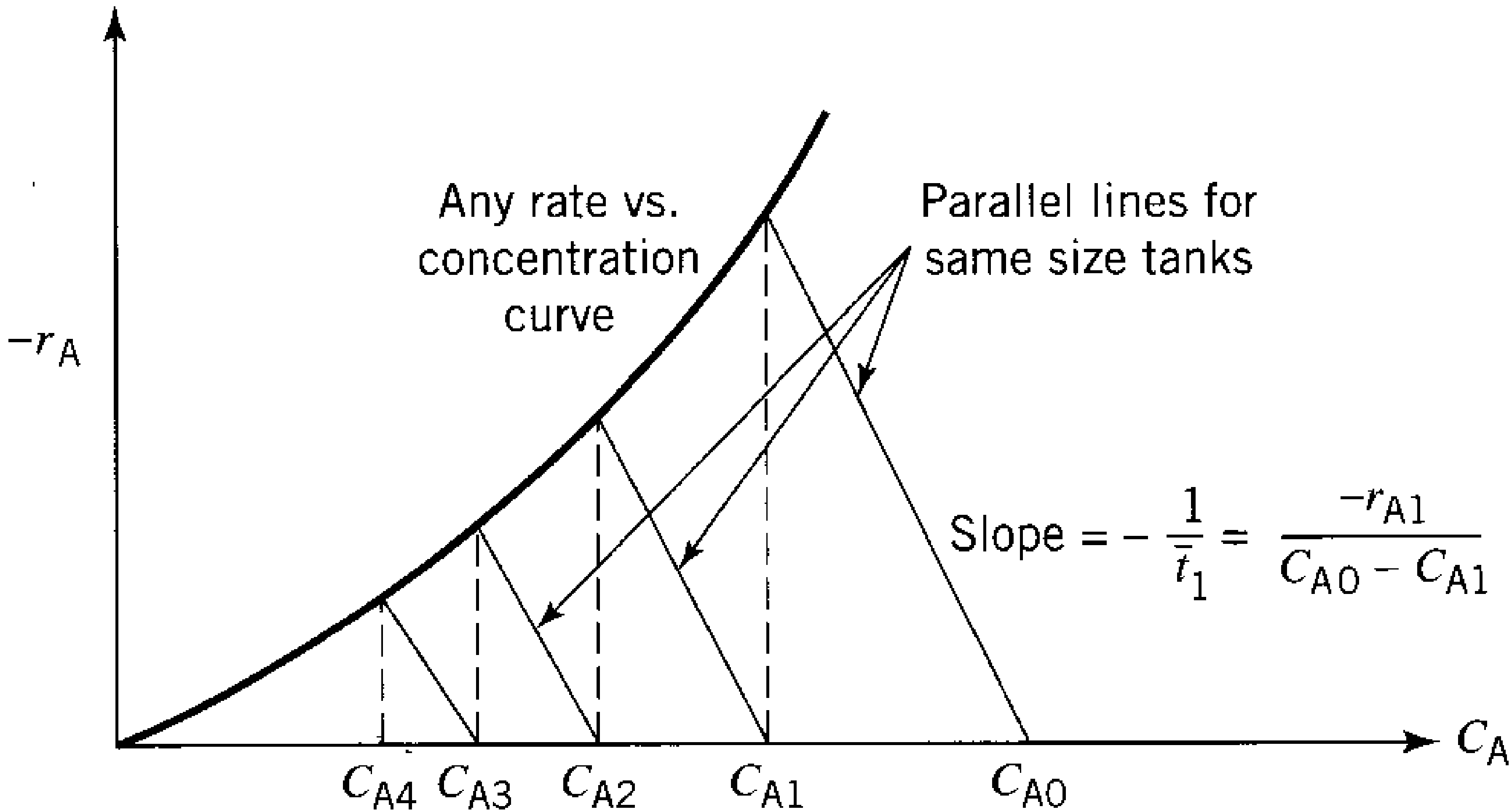


Figure 14.8 Graphical method of evaluating the performance of N tanks in series for any kinetics.

- Chemical Conversion of Macrofluid

$$\frac{C_A}{C_{A0}} = \frac{N^N}{(N-1)!\,\bar{t}^N}\int_0^\infty \left(\frac{C_A}{C_{A0}}\right)_{batch} t^{N-1} e^{-\frac{tN}{\bar{t}}}\,dt$$

$$\frac{C_A}{C_{A0}} = \int_0^\infty \left(\frac{C_A}{C_{A0}}\right)_{batch} E\,dt$$

$$\bar{t}E = \left(\frac{t}{\bar{t}}\right)^{N-1}\frac{N^N}{(N-1)!}e^{-\frac{tN}{\bar{t}}} \qquad E = \frac{1}{\bar{t}}\left(\frac{t}{\bar{t}}\right)^{N-1}\frac{N^N}{(N-1)!}e^{-\frac{tN}{\bar{t}}}$$

$$\frac{C_A}{C_{A0}} = \int_0^\infty \left(\frac{C_A}{C_{A0}}\right)_{batch} \frac{1}{\bar{t}}\left(\frac{t}{\bar{t}}\right)^{N-1}\frac{N^N}{(N-1)!}e^{-\frac{tN}{\bar{t}}}\,dt$$

$$= \frac{N^N}{(N-1)!\,\bar{t}^N}\int_0^\infty \left(\frac{C_A}{C_{A0}}\right)_{batch} t^{N-1} e^{-\frac{tN}{\bar{t}}}\,dt$$

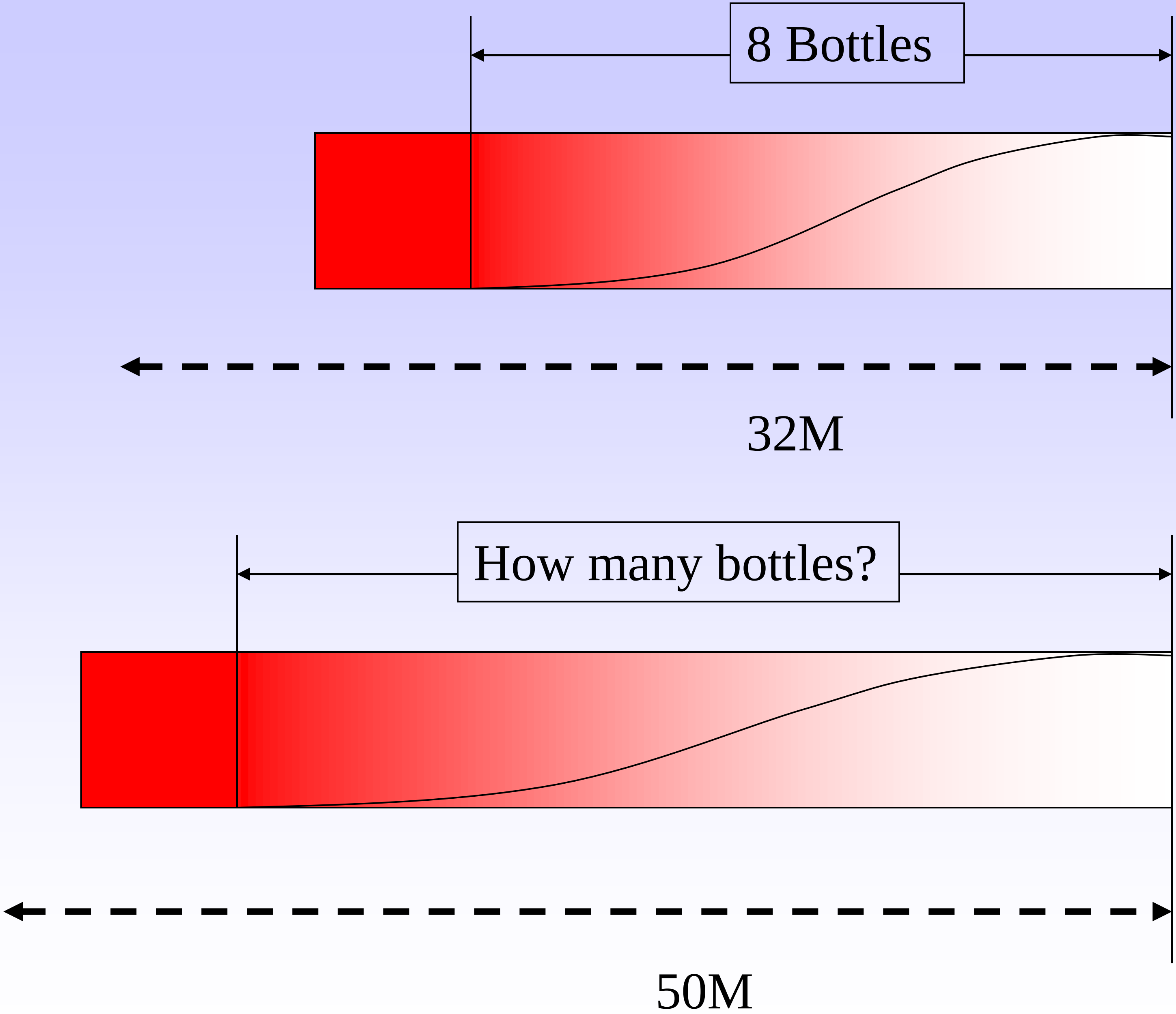
EXAMPLE 14.1 MODIFICATIONS TO A WINERY

A small diameter pipe 32 m long runs from the fermentation room of a winery to the bottle filling cellar. Sometimes red wine is pumped through the pipe, sometimes white, and whenever the switch is made from one to the other a small amount of "house blend" rosé is produced (8 bottles). Because of some construction in the winery the pipeline length will have to be increased to 50 m. For the same flow rate of wine, how many bottles of rosé may we now expect to get each time we switch the flow?

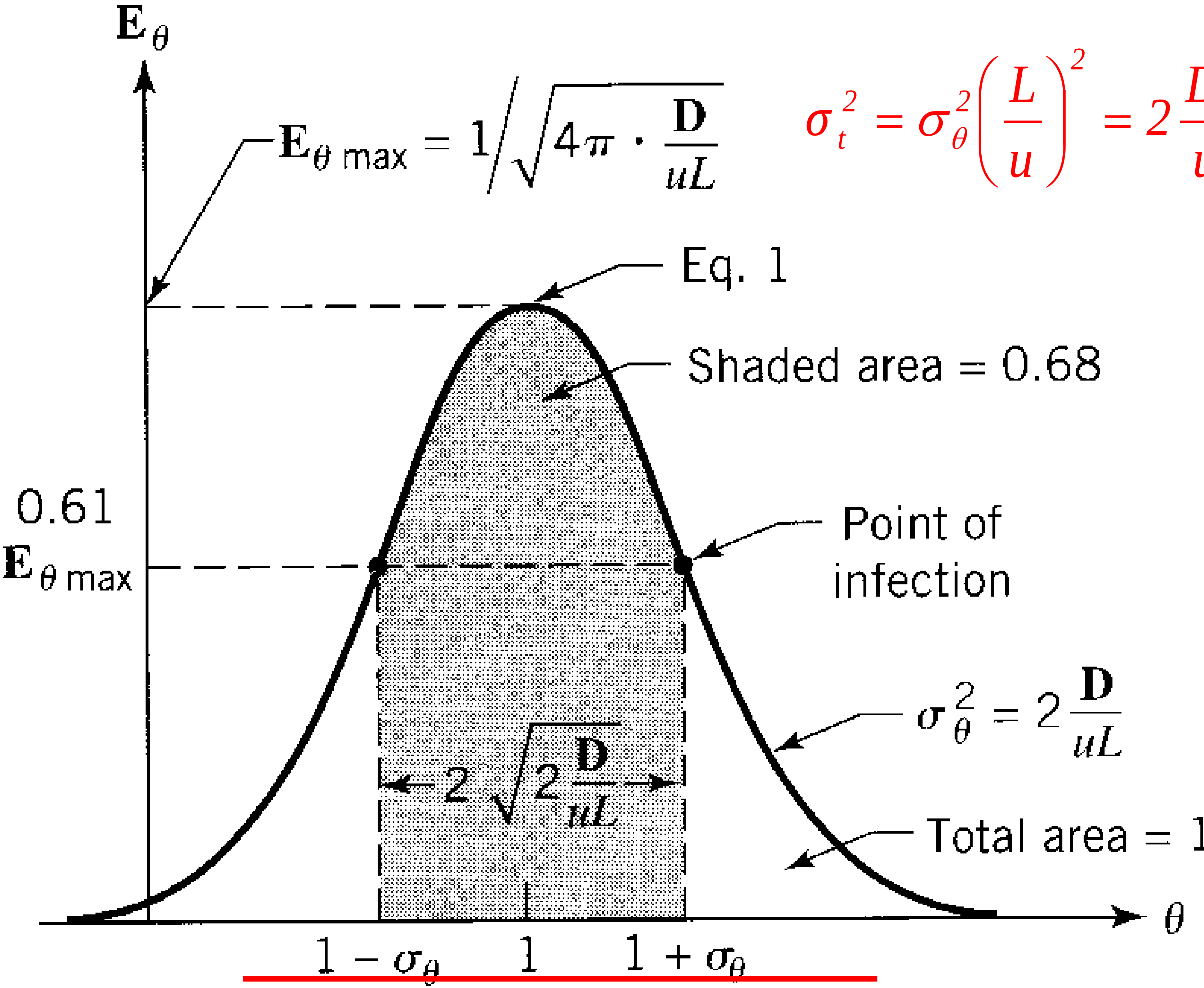
SOLUTION

Figure E14.1 sketches the problem. Let the number of bottles, the spread, be related to σ.

Original:	$L_1 = 32$ m	$\sigma_1 = 8$	$\sigma_1^2 = 64$
Longer pipe:	$L_2 = 50$ m	$\sigma_2 = ?$	$\sigma_2^2 = ?$


8 Bottles
32M
How many bottles?
50M

$$\sigma_t^2 = \sigma_\theta^2 \left(\frac{L}{u}\right)^2 = 2\frac{DL}{u^3}$$



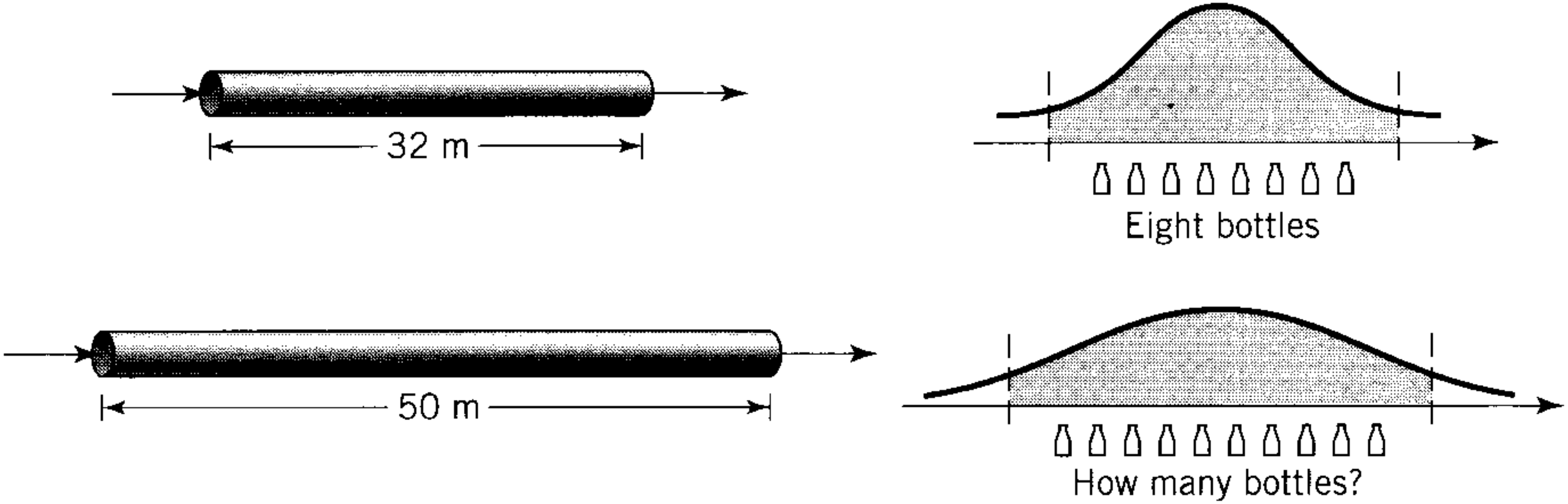


Figure E14.1

But for small deviations from plug flow, from Eq. 3 $\sigma^2 \propto N$ or $\sigma^2 \propto$ L.

$$\therefore \frac{\sigma_2^2}{\sigma_1^2} = \frac{L_2}{L_1} = \frac{50}{32}$$

$$\sigma_t^2 = \sigma_\theta^2 \left(\frac{L}{u}\right)^2 = 2\frac{DL}{u^3}$$

$$\therefore \sigma_2^2 = \frac{50}{32}(64) = 100$$

$\therefore \sigma_2 = 10$ . . . or we can expect 10 bottles of vin rosé

EXAMPLE 14.2 A FABLE ON RIVER POLLUTION

Last spring our office received complaints of a large fish kill along the Ohio River, indicating that someone had discharged highly toxic material into the river. Our water monitoring stations at Cincinnati and Portsmouth, Ohio (119 miles apart), report that a large slug of phenol is moving down the river, and we strongly suspect that this is the cause of the pollution. The slug took about 10.5 hours to pass the Portsmouth monitoring station, and its concentration peaked at 8:00 A.M. Monday. About 26 hours later the slug peaked at Cincinnati, taking 14 hours to pass this monitoring station.

Phenol is used at a number of locations on the Ohio River, and their distance upriver from Cincinnati are as follows:

Ashland, KY—150 miles upstream	Marietta, OH—303
Huntington, WV—168	Wheeling, WV—385
Pomeroy, OH—222	Steubenville, OH—425
Parkersburg, WV—290	Pittsburgh, PA—500

What can you say about the probable pollution source?

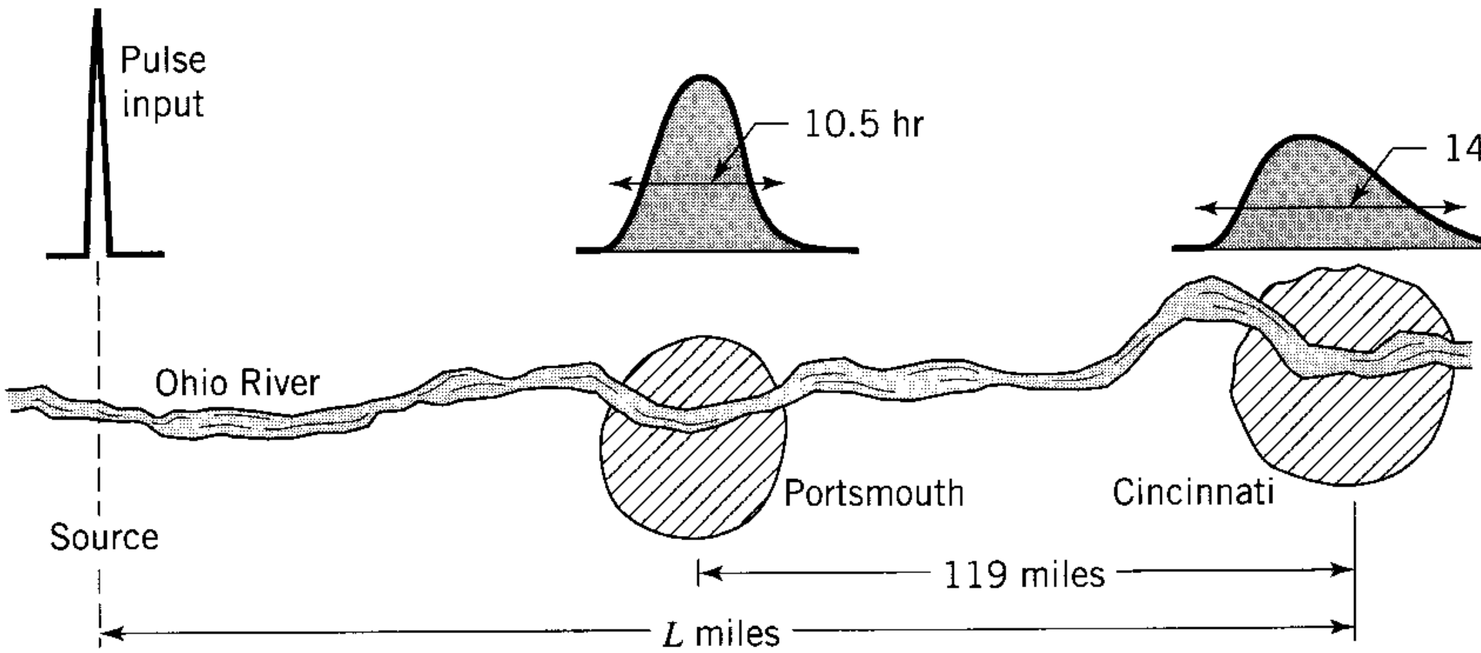


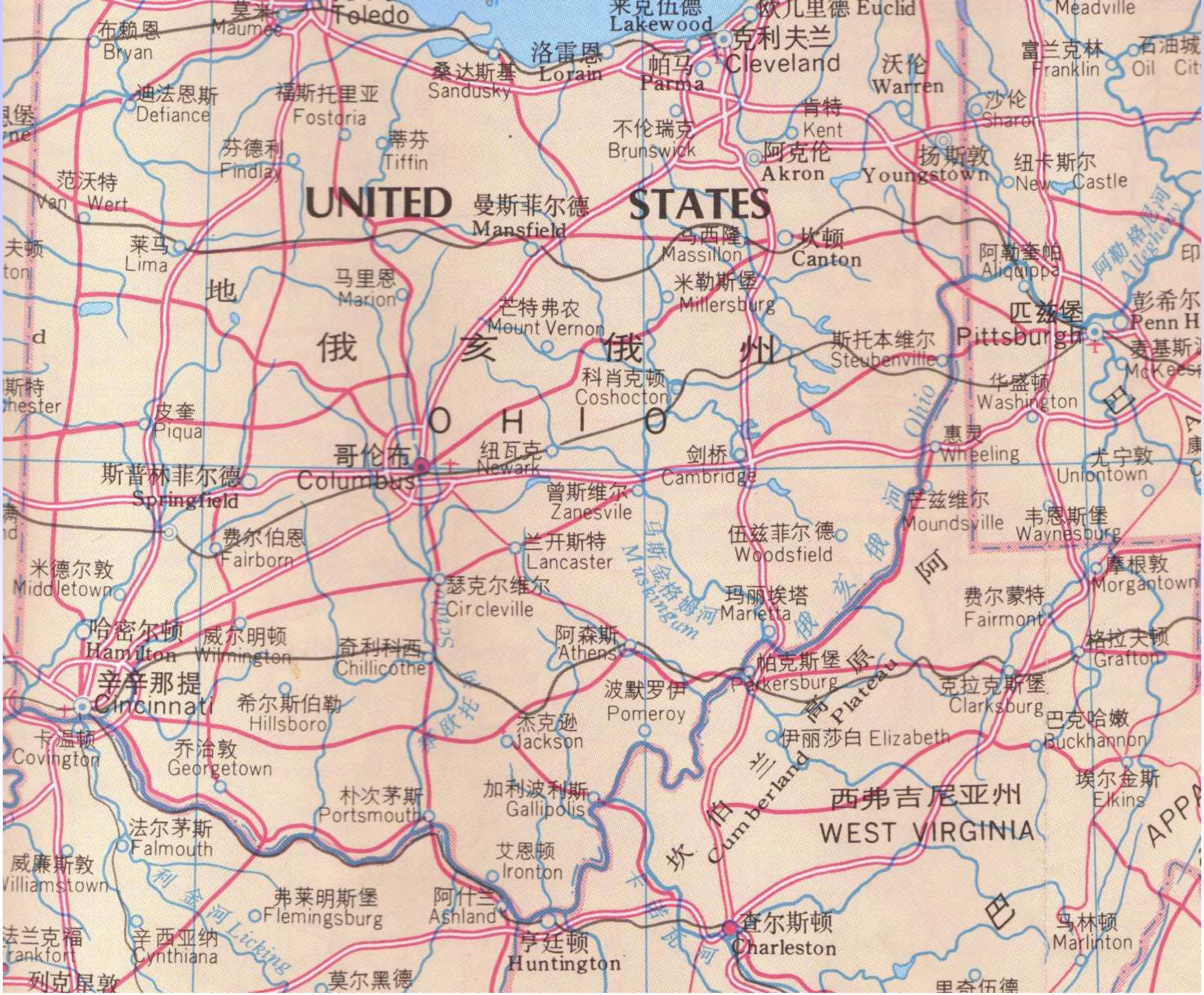
Figure E14.2

SOLUTION

Let us first sketch what is known, as shown in Fig. E14.2. To start, assume that a perfect pulse is injected. Then according to any reasonable flow model, either dispersion or tanks-in-series, we have

$$\sigma^2_{\text{tracer curve}} \propto \begin{pmatrix}\text{distance from} \\ \text{point of origin}\end{pmatrix} \quad \text{or} \quad \begin{pmatrix}\text{spread of} \\ \text{curve}\end{pmatrix} \propto \sqrt{\begin{matrix}\text{distance from} \\ \text{origin}\end{matrix}}$$

$$\sigma_t^2 = 2\frac{DL}{u^3}$$


UNITED STATES
俄亥俄州
OHIO
西弗吉尼亚州
WEST VIRGINIA
坎伯兰高原 Cumberland Plateau
阿巴
莱克伍德 Lakewood
欧几里德 Euclid
Toledo
莫米 Maumee
布赖恩 Bryan
迪法恩斯 Defiance
福斯托里亚 Fostoria
桑达斯基 Sandusky
洛雷恩 Lorain
帕马 Parma
克利夫兰 Cleveland
沃伦 Warren
富兰克林 Franklin
石油城 Oil City
Meadville
沙伦 Sharon
肯特 Kent
不伦瑞克 Brunswick
阿克伦 Akron
扬斯敦 Youngstown
纽卡斯尔 New Castle
芬德利 Findlay
蒂芬 Tiffin
范沃特 Van Wert
曼斯菲尔德 Mansfield
马西隆 Massillon
坎顿 Canton
阿勒奎帕 Aliquippa
阿勒格尼河 Allegheny
莱马 Lima
马里恩 Marion
米勒斯堡 Millersburg
芒特弗农 Mount Vernon
匹兹堡 Pittsburgh
彭希尔 Penn H
麦基斯 McKees
斯托本维尔 Steubenville
科肖克顿 Coshocton
华盛顿 Washington
皮奎 Piqua
哥伦布 Columbus
纽瓦克 Newark
剑桥 Cambridge
惠灵 Wheeling
尤宁敦 Uniontown
斯普林菲尔德 Springfield
曾斯维尔 Zanesvile
芒兹维尔 Moundsville
韦恩斯堡 Waynesburg
费尔伯恩 Fairborn
兰开斯特 Lancaster
伍兹菲尔德 Woodsfield
马斯金格姆河 Muskingum
米德尔敦 Middletown
瑟克尔维尔 Circleville
玛丽埃塔 Marietta
俄亥俄河 Ohio
摩根敦 Morgantown
费尔蒙特 Fairmont
哈密尔顿 Hamilton
威尔明顿 Wilmington
奇利科西 Chillicothe
Scioto
阿森斯 Athens
帕克斯堡 Parkersburg
格拉夫顿 Grafton
辛辛那提 Cincinnati
希尔斯伯勒 Hillsboro
波默罗伊 Pomeroy
克拉克斯堡 Clarksburg
杰克逊 Jackson
伊丽莎白 Elizabeth
巴克哈嫩 Buckhannon
卡温顿 Covington
乔治敦 Georgetown
朴次茅斯 Portsmouth
加利波利斯 Gallipolis
埃尔金斯 Elkins
法尔茅斯 Falmouth
艾恩顿 Ironton
威廉斯敦 Williamstown
弗莱明斯堡 Flemingsburg
阿什兰 Ashland
利金河 Licking
卡诺瓦河
法兰克福 Frankfort
辛西亚纳 Cynthiana
亨廷顿 Huntington
查尔斯顿 Charleston
马林顿 Marlinton
莫尔黑德
列克星敦
APPA

$$\left.\begin{aligned} &\therefore \text{ from Cincinnati:} && 14 = k\,L^{1/2} \\ &\therefore \text{ from Portsmouth:} && 10.5 = k(L - 119)^{1/2} \end{aligned}\right\}$$

Dividing one by the other gives

$$\frac{14}{10.5} = \sqrt{\frac{L}{L - 119}} \qquad \text{. . . from which } \underline{\underline{L = 272 \text{ miles}}}$$

Comment. Since the dumping of the toxic phenol may not have occurred instantaneously, any location where $L \leq 272$ miles is suspect, or

$$\left.\begin{array}{l} \underline{\underline{\text{Ashland}}} \\ \underline{\underline{\text{Huntington}}} \\ \underline{\underline{\text{Pomeroy}}} \end{array}\right\} \leftarrow$$

This solution assumes that different stretches of the Ohio River have the same flow and dispersion characteristics (reasonable), and that no suspect tributary joins the Ohio within 272 miles of Cincinnati. This is a poor assumption . . . check a map for the location of Charleston, WV, on the Kanawah River.

EXAMPLE 14.3 FLOW MODELS FROM RTD CURVES

Let us develop a tanks-in-series model to fit the RTD shown in Fig. E14.3*a*.

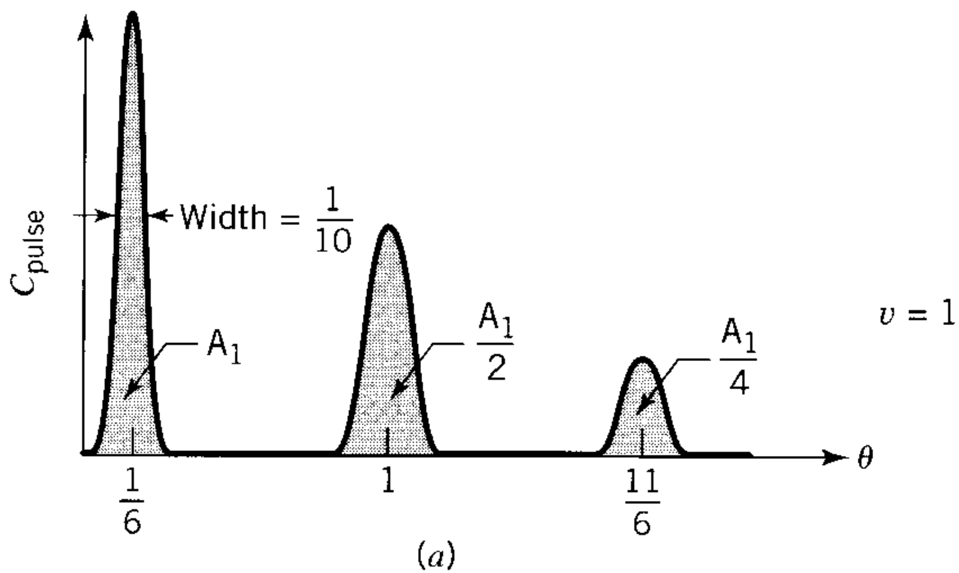
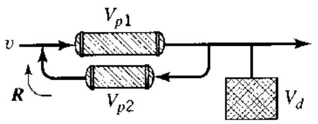


Figure E14.3*a*

SOLUTION

As a first approximation, assume that all the tracer curves are ideal pulses. We will later relax this assumption. Next notice that the first pulse appears early. This suggests a model as shown in Fig. E14.3*b*, where $v = 1$ and $V_1 + V_2 + V_d = 1$. In Chapter 12 we see the characteristics of this model, so let us fit it. Also it should be mentioned that we have a number of approaches. Here is one:



- Look at the ratio of areas of the first two peaks

See P285

$$\frac{A_2}{A_1} = \frac{1}{2} = \frac{R}{R+1} \qquad \cdots R = 1$$

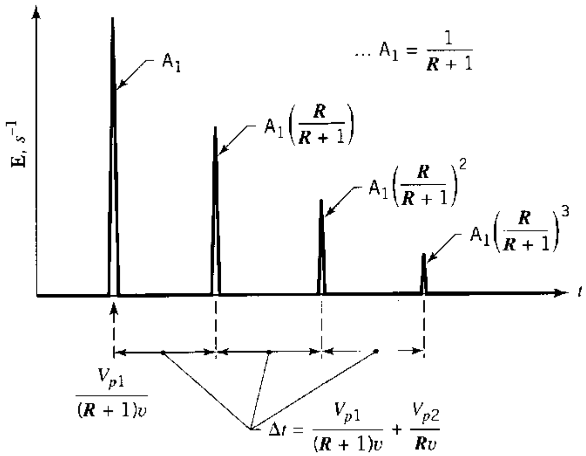
- From the location of the first peak

$$\frac{V_1}{(R+1)v} = \frac{V_1}{(1+1)} = \frac{1}{6} \qquad \cdots V_1 = \frac{1}{3}$$

- From the time between peaks

$$\Delta\theta = \frac{5}{6} = \frac{(1/3)}{(1+1)1} + \frac{V_2}{1(1)} \qquad \cdots V_2 = \frac{2}{3}$$

Since $V_1 + V_2$ add up to 1, there is no dead volume, so at this point our model reduces to Fig. E14.3*c*. Now relax the plug flow assumption and adopt the tanks-in-series model. From Fig. 14.3



For the first peak, and for any other peaks,

$\Delta\theta$ is the width between infection points.

θ_{max} is the θ value when E_θ gets the maximum

$$\frac{\Delta\theta}{\theta_{max}} = \frac{1/10}{1/6} = \frac{2}{\sqrt{N-1}} \qquad \cdots N = 12$$

So our model finally is shown in Fig. E14.3*d*.

Extension :

what is the width of the second peak?

$$\frac{\Delta\theta}{\theta_{max}} = \frac{2}{\sqrt{N-1}} \qquad \frac{\Delta\theta}{1} = \frac{2}{\sqrt{(12+24)-1}} \qquad \Delta\theta = 0.34$$

And the third?

$$\frac{\Delta\theta}{11/6} = \frac{2}{\sqrt{(12+24)-1}} \qquad \Delta\theta = 0.62$$

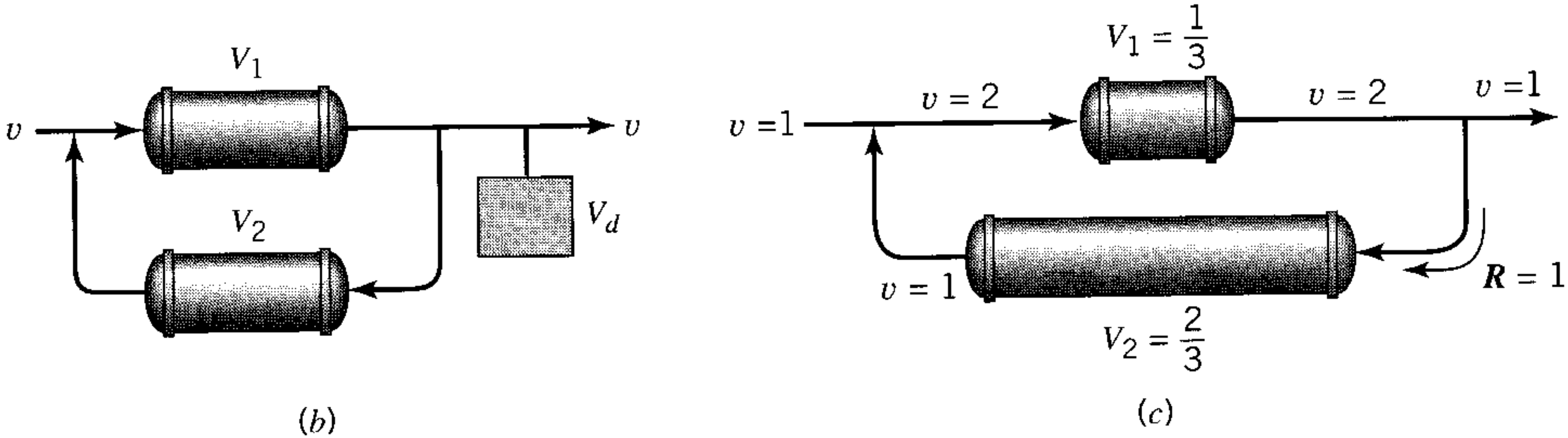


Figure E14.3*b* and *c*

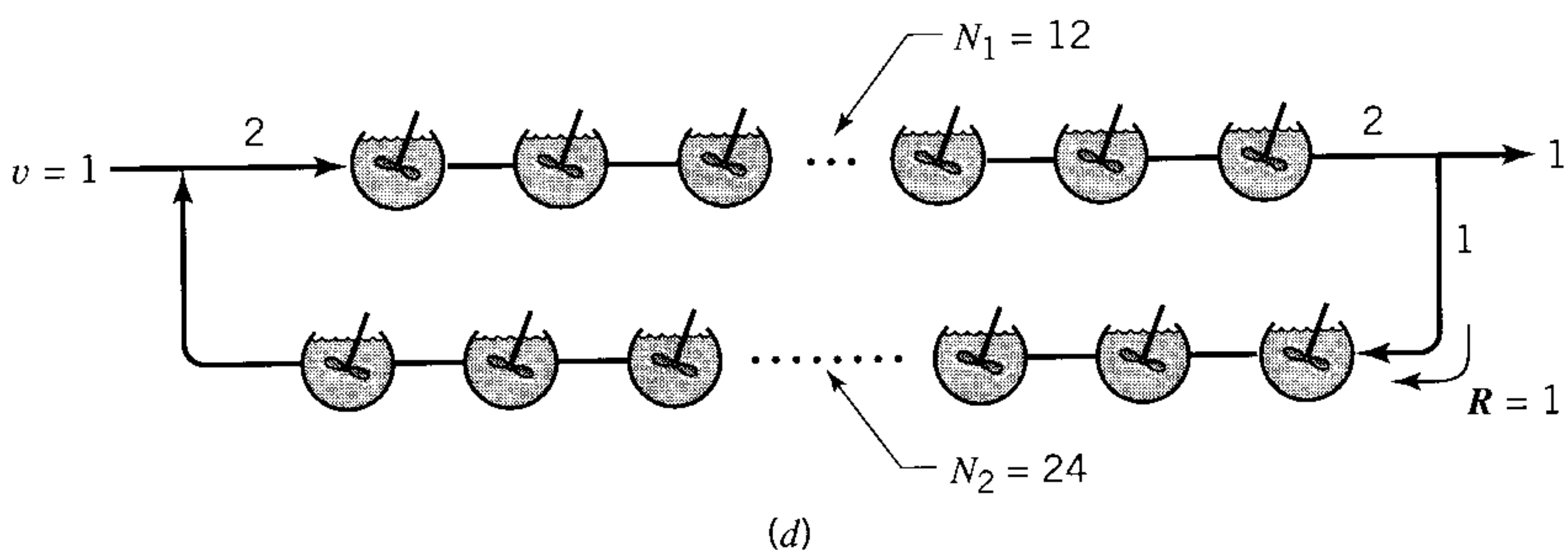
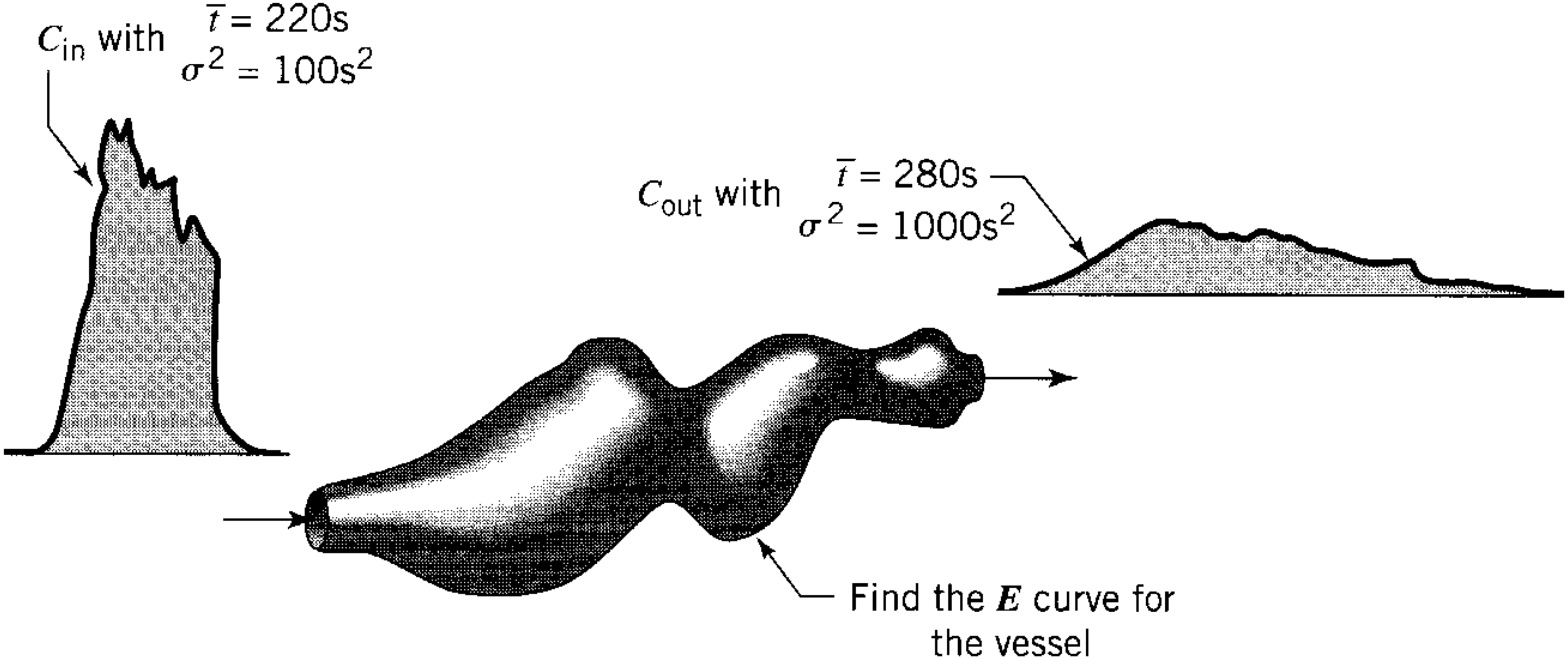


Figure E14.3*d*

EXAMPLE 14.4 FINDING THE VESSEL E CURVE USING A SLOPPY TRACER INPUT

Given C_{in} and C_{out} as well as the location and spread of these tracer curves, as shown in Fig. E14.4*a* estimate the vessel **E** curve. We suspect that the tanks-in-series model reasonably represents the flow in the vessel.



(*a*)

Figure E14.4*a*

SOLUTION

From Fig. E14.4*a* we have, for the vessel,

$$\Delta\bar{t} = 280 - 220 = 60 \text{ s}$$

$$\Delta(\sigma^2) = 1000 - 100 = 900 \text{ s}^2$$

Equation 3 represents the tanks-in-series model and gives

$$\sigma^2 = \frac{\bar{t}^2}{N} \qquad N = \frac{(\Delta\bar{t})^2}{\Delta(\sigma^2)} = \frac{60^2}{900} = 4 \text{ tanks}$$

So from Eq. 3a, for N tanks-in-series we have

$$\mathbf{E} = \frac{t^{N-1}}{\bar{t}^N} \cdot \frac{N^N}{(N-1)!} e^{-tN/\bar{t}}$$

and for $N = 4$

$$\mathbf{E} = \frac{t^3}{60^4} \cdot \frac{4^4}{3 \times 2} e^{-4t/60}$$

$$\underline{\underline{\mathbf{E} = 3.2922 \times 10^{-6}\, t^3\, e^{-0.0667t}}}$$

- N is not necessarily an integer.

  The value of (decimal)! could be found by Γ function

$$\Gamma(z) = \int_0^\infty u^{z-1} e^{-u} du$$

$$\Gamma(n+1) = n!$$

$$\Gamma(x) = \frac{\Gamma(x+1)}{x}, \qquad \Gamma(x) = (x-1)\Gamma(x-1)$$

$\Gamma(1 \le x < 2)$ may be found in a handbook

Figure E14.4*b* shows the shape of this **E** curve.

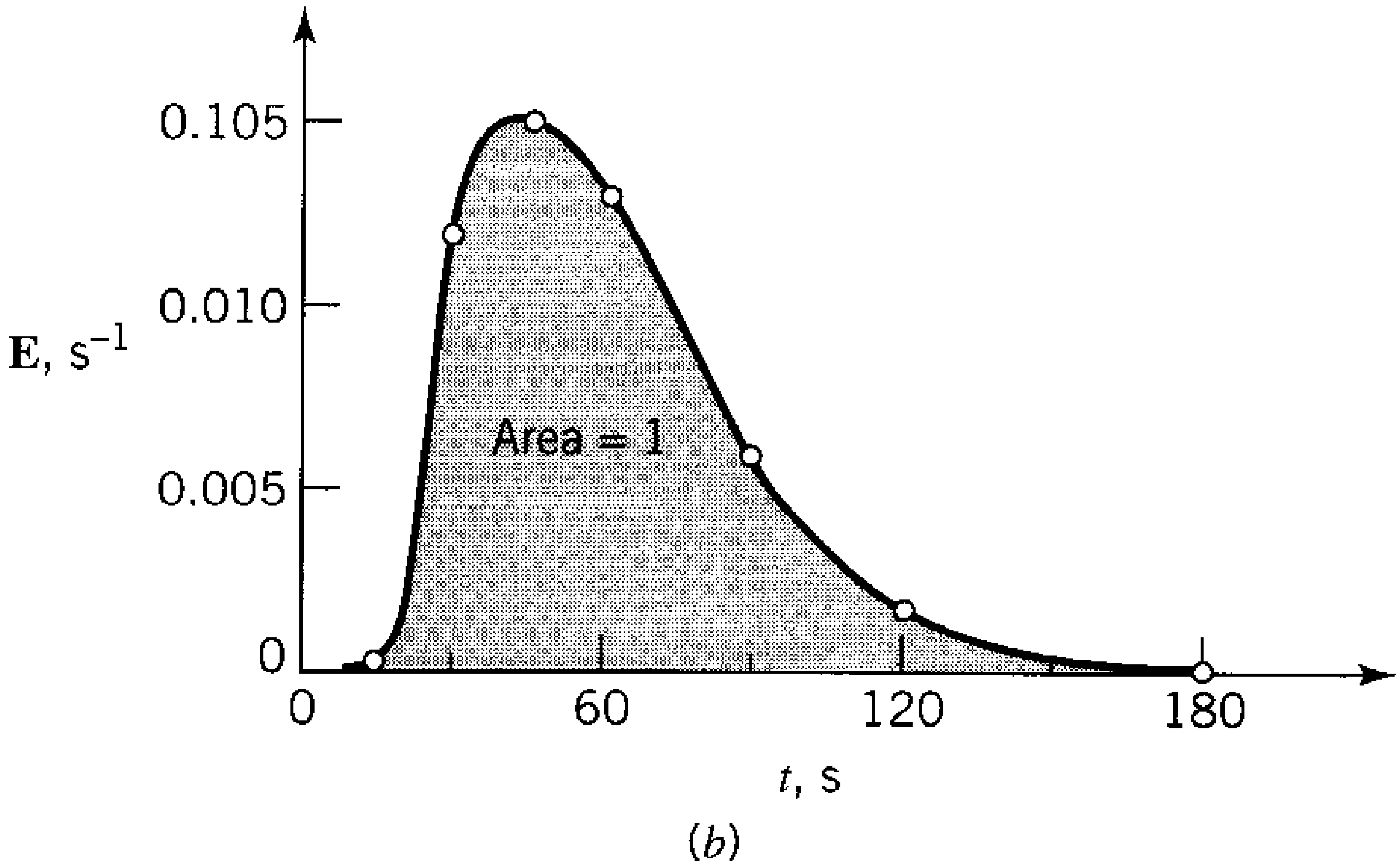


Figure E14.4*b*