# Chapter two Conservation laws of fluid motion and boundary conditions

In this chapter we develop the mathematical basis for a comprehensive general-purpose model of fluid flow and heat transfer from the basic principles of conservation of mass, momentum and energy. This leads to the governing equations of fluid flow and a discussion of the necessary auxiliary conditions – initial and boundary conditions. The main issues covered in this context are:

- Derivation of the system of partial differential equations (PDEs) that govern flows in Cartesian ($x$, $y$, $z$) co-ordinates
- Thermodynamic equations of state
- Newtonian model of viscous stresses leading to the Navier–Stokes equations
- Commonalities between the governing PDEs and the definition of the transport equation
- Integrated forms of the transport equation over a finite time interval and a finite control volume
- Classification of physical behaviours into three categories: elliptic, parabolic and hyperbolic
- Appropriate boundary conditions for each category
- Classification of fluid flows
- Auxiliary conditions for viscous fluid flows
- Problems with boundary condition specification in high Reynolds number and high Mach number flows

## 2.1 Governing equations of fluid flow and heat transfer

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The governing equations of fluid flow represent mathematical statements of the **conservation laws of physics**:

- The mass of a fluid is conserved
- The rate of change of momentum equals the sum of the forces on a fluid particle (Newton's second law)
- The rate of change of energy is equal to the sum of the rate of heat addition to and the rate of work done on a fluid particle (first law of thermodynamics)

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The fluid will be regarded as a continuum. For the analysis of fluid flows at macroscopic length scales (say 1 μm and larger) the molecular structure

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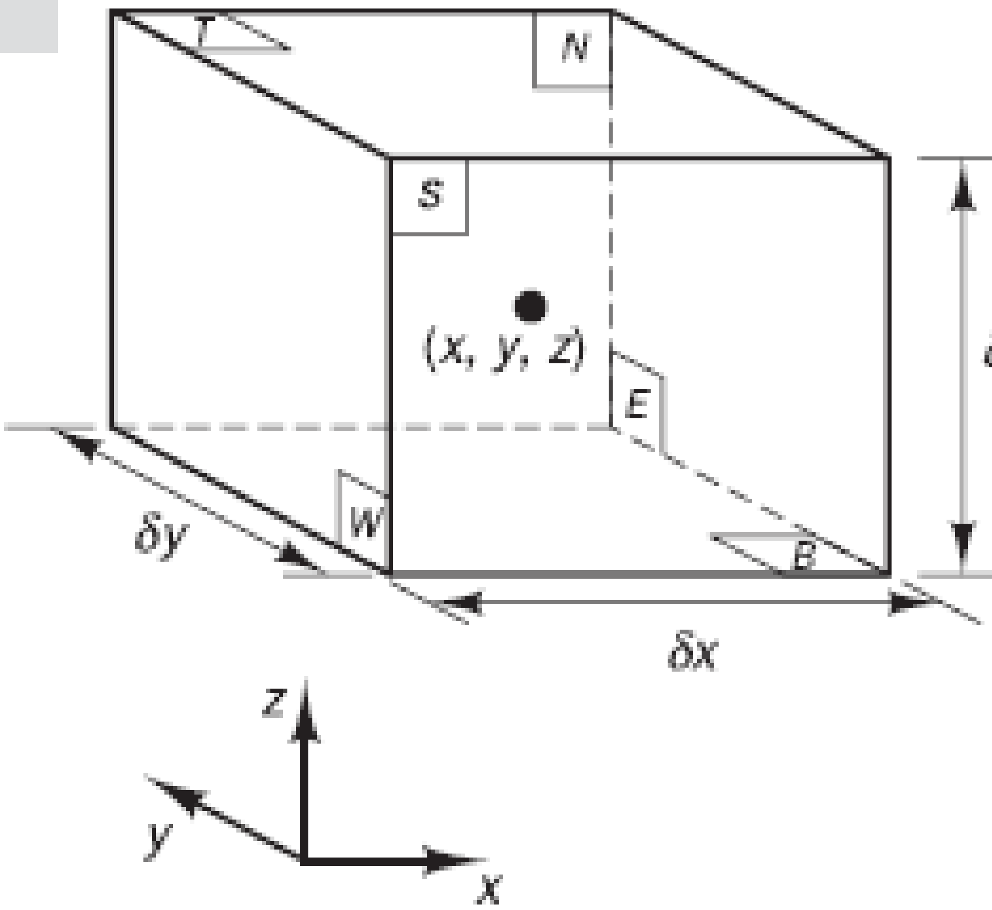
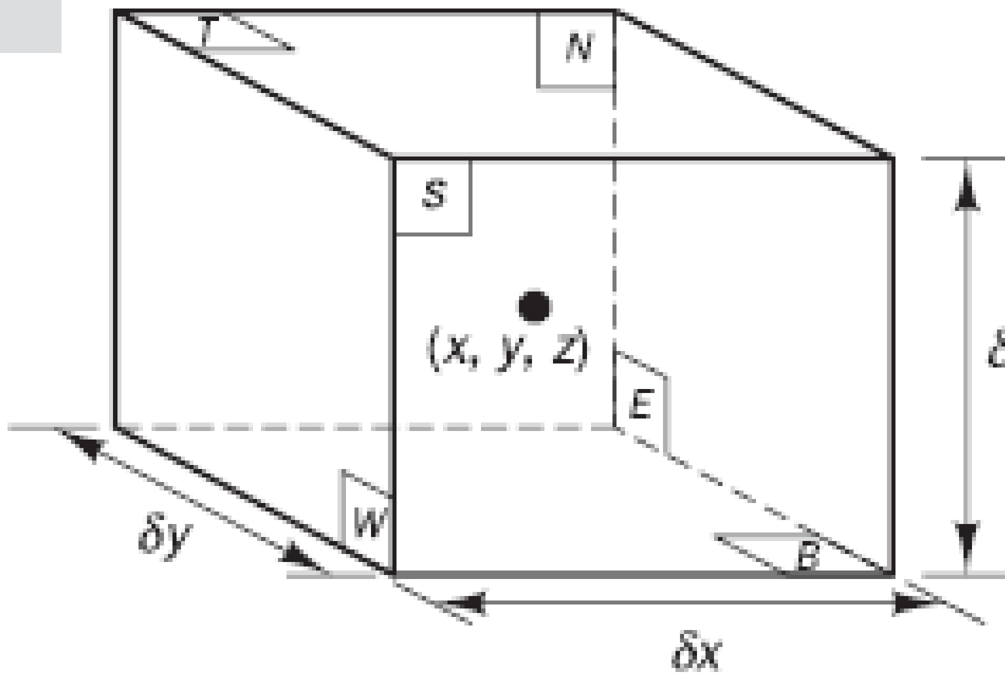


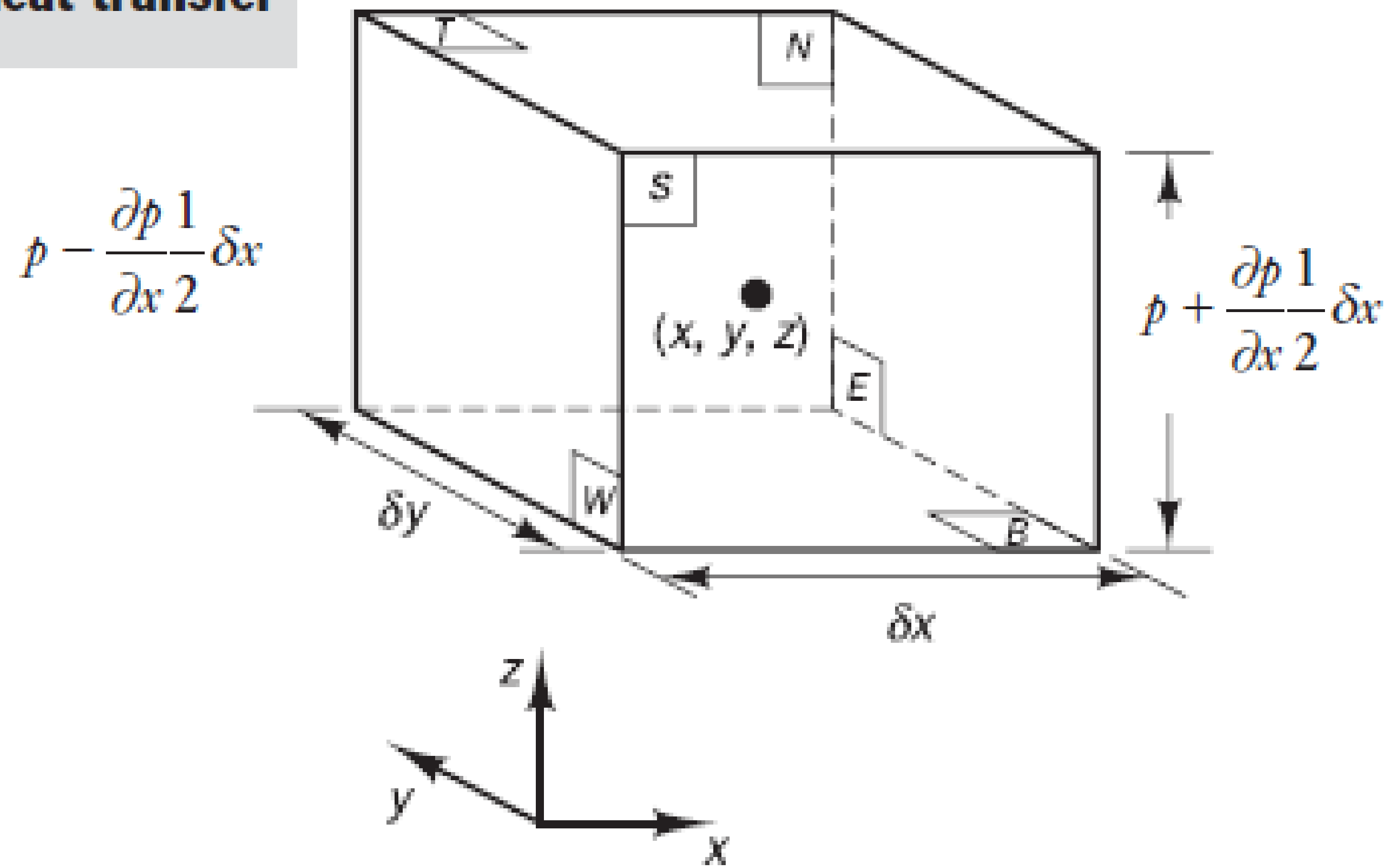

**Figure 2.1** Fluid element for conservation laws

**2.1** **Governing equations of fluid flow and heat transfer**




All fluid properties are functions of space and time so we would strictly need to write $\rho(x, y, z, t)$, $p(x, y, z, t)$, $T(x, y, z, t)$ and $\mathbf{u}(x, y, z, t)$ for the density, pressure, temperature and the velocity vector respectively. To avoid

# 2.1 Governing equations of fluid flow and heat transfer



# 2.1 Governing equations of fluid flow and heat transfer

## 2.1.1 Mass conservation in three dimensions

The first step in the derivation of the mass conservation equation is to write down a mass balance for the fluid element:

| | | |
|---|---|---|
| Rate of increase of mass in fluid element | = | Net rate of flow of mass into fluid element |

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The rate of increase of mass in the fluid element is

$$\frac{\partial}{\partial t}(\rho\delta x\delta y\delta z) = \frac{\partial\rho}{\partial t}\delta x\delta y\delta z$$

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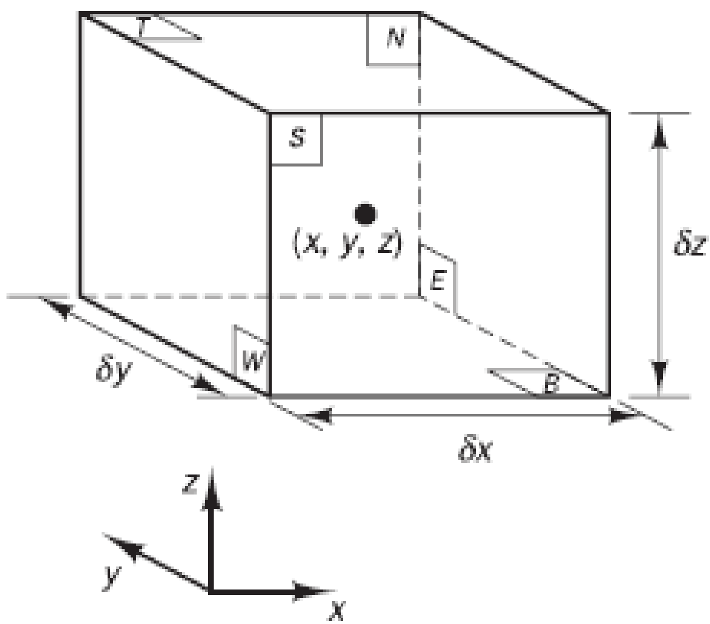

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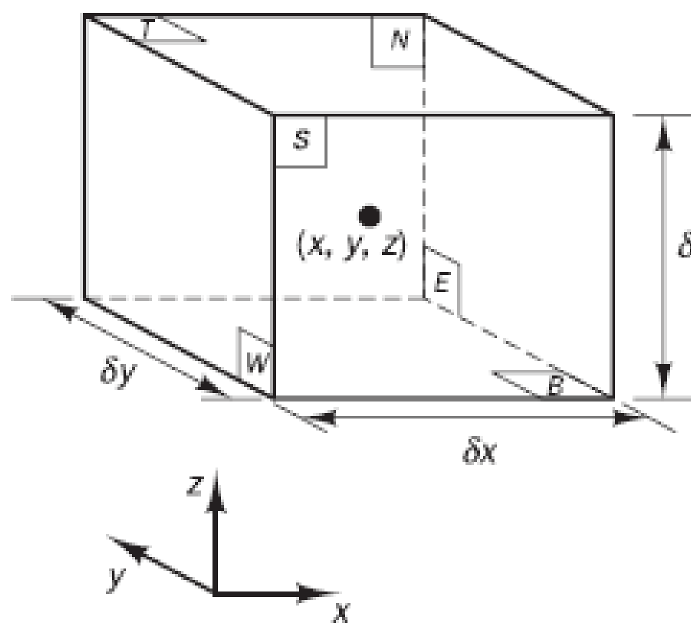

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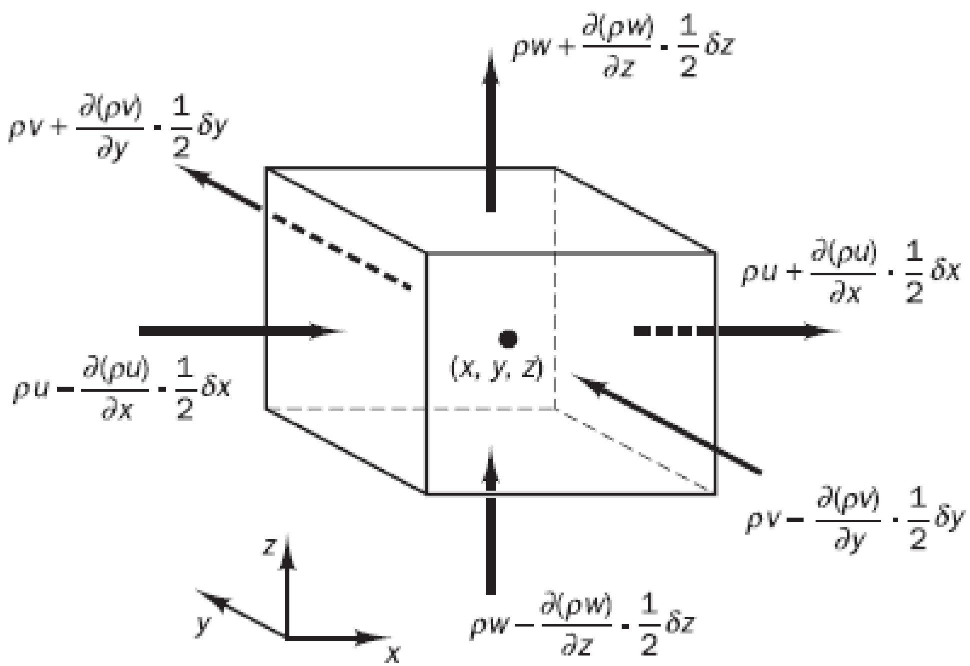


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|---|---|---|

$$\left(\rho u - \frac{\partial(\rho u)}{\partial x}\frac{1}{2}\delta x\right)\delta y\delta z - \left(\rho u + \frac{\partial(\rho u)}{\partial x}\frac{1}{2}\delta x\right)\delta y\delta z$$

$$+ \left(\rho v - \frac{\partial(\rho v)}{\partial y}\frac{1}{2}\delta y\right)\delta x\delta z - \left(\rho v + \frac{\partial(\rho v)}{\partial y}\frac{1}{2}\delta y\right)\delta x\delta z$$

$$+ \left(\rho w - \frac{\partial(\rho w)}{\partial z}\frac{1}{2}\delta z\right)\delta x\delta y - \left(\rho w + \frac{\partial(\rho w)}{\partial z}\frac{1}{2}\delta z\right)\delta x\delta y \qquad (2.2)$$

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$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0$$

or in more compact vector notation

$$\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \mathbf{u}) = 0$$

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) \equiv \nabla \cdot (\rho \mathbf{V}) \tag{4.5}$$

so that the compact form of the continuity relation is

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0 \tag{4.6}$$

In this vector form the equation is still quite general and can readily be converted to other than cartesian coordinate systems.

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0 \tag{2.3}$$

or in more compact vector notation

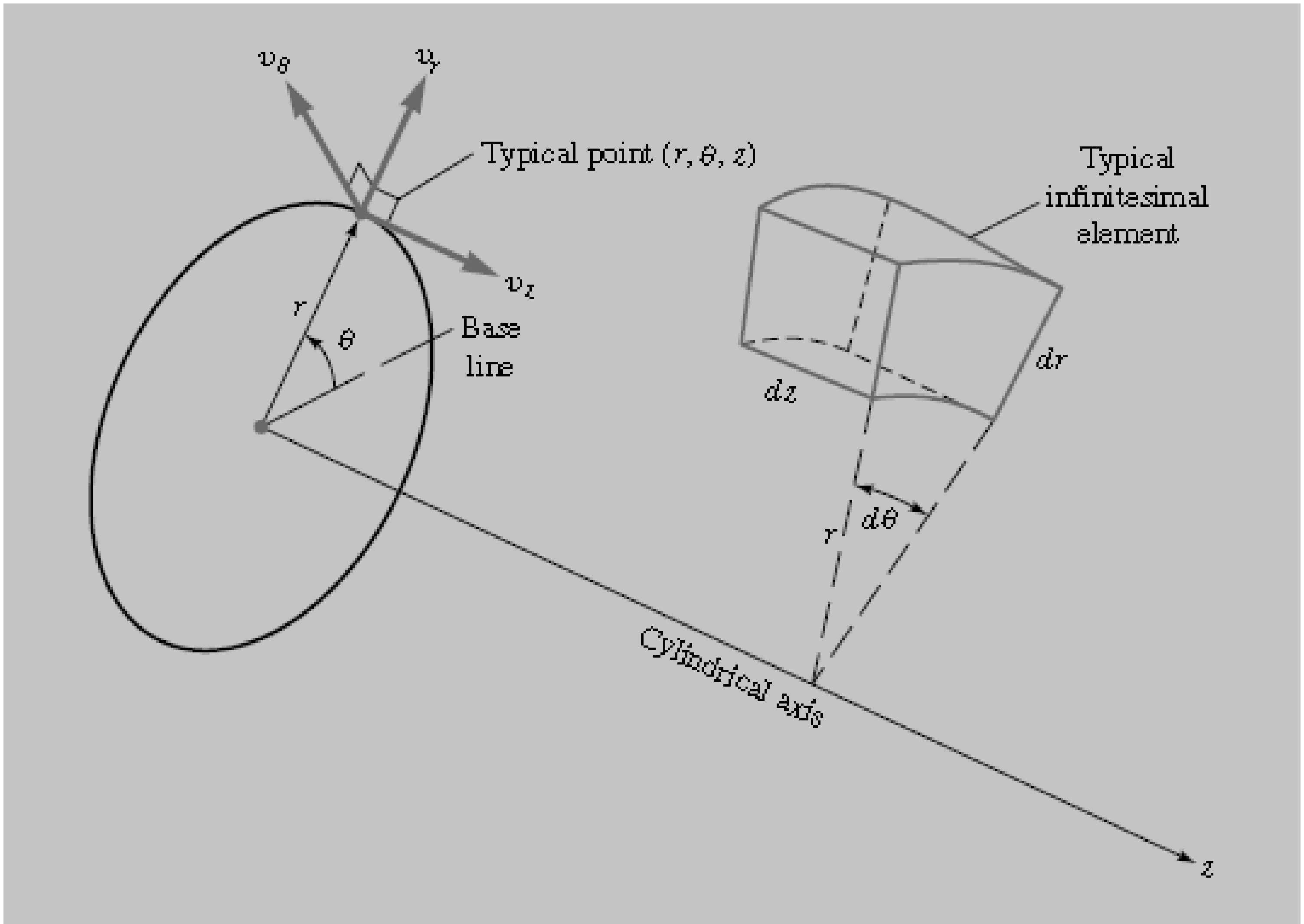
$$\boxed{\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \mathbf{u}) = 0} \tag{2.4}$$

Equation (2.4) is the **unsteady, three-dimensional mass conservation or continuity equation** at a point in a **compressible fluid**. The first term

- Using vector notation can be written as follows

$$\underbrace{\frac{\partial \rho}{\partial t}}_{\substack{rate\\of\\increase\\of\ mass\\per\ unit\\volume}} = -\underbrace{\left(\nabla . \rho v\right)}_{\substack{net\ rate\\of\ mass\\addition\\per\ unit\\volume\\by\\convection}}$$

# Cylindrical Polar Coordinates



$$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}=0 \tag{2.3}$$

or in more compact vector notation

$$\boxed{\frac{\partial \rho}{\partial t}+\operatorname{div}(\rho \mathbf{u})=0} \tag{2.4}$$

Equation (2.4) is the **unsteady, three-dimensional mass conservation or continuity equation** at a point in a **compressible fluid**. The first term

For an **incompressible fluid** (i.e. a liquid) the density $\rho$ is constant and equation (2.4) becomes

$$\operatorname{div} \mathbf{u}=0 \tag{2.5}$$

or in longhand notation

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \tag{2.6}$$

### 2.1.2 Rates of change following a fluid particle and for a fluid element

In Sec. 1.7 we established the cartesian vector form of a velocity field that varies in space and time:

$$\mathbf{V}(\mathbf{r}, t) = \mathbf{i}u(x, y, z, t) + \mathbf{j}\upsilon(x, y, z, t) + \mathbf{k}w(x, y, z, t) \tag{1.4}$$

$$\mathbf{a} = \frac{d\mathbf{V}}{dt} = \mathbf{i}\frac{du}{dt} + \mathbf{j}\frac{d\upsilon}{dt} + \mathbf{k}\frac{dw}{dt}$$

Since each scalar component ($u$, $\upsilon$, $w$) is a function of the four variables ($x$, $y$, $z$, $t$), we use the chain rule to obtain each scalar time derivative. For example,

$$\frac{du(x, y, z, t)}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

But, by definition, $dx/dt$ is the local velocity component $u$, and $dy/dt = \upsilon$, and $dz/dt = w$. The total time derivative of $u$ may thus be written as follows, with exactly similar expressions for the time derivatives of $\upsilon$ and $w$:

$$a_x = \frac{du}{dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \upsilon\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = \frac{\partial u}{\partial t} + (\mathbf{V}\cdot\nabla)u$$

$$a_y = \frac{d\upsilon}{dt} = \frac{\partial \upsilon}{\partial t} + u\frac{\partial \upsilon}{\partial x} + \upsilon\frac{\partial \upsilon}{\partial y} + w\frac{\partial \upsilon}{\partial z} = \frac{\partial \upsilon}{\partial t} + (\mathbf{V}\cdot\nabla)\upsilon \tag{4.1}$$

$$a_z = \frac{dw}{dt} = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + \upsilon\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = \frac{\partial w}{\partial t} + (\mathbf{V}\cdot\nabla)w$$

Summing these into a vector, we obtain the total acceleration:

$$\mathbf{a} = \frac{d\mathbf{V}}{dt} = \underbrace{\frac{\partial \mathbf{V}}{\partial t}}_{\text{Local}} + \underbrace{\left(u\frac{\partial \mathbf{V}}{\partial x} + \upsilon\frac{\partial \mathbf{V}}{\partial y} + w\frac{\partial \mathbf{V}}{\partial z}\right)}_{\text{Convective}} = \frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\cdot\boldsymbol{\nabla})\mathbf{V} \tag{4.2}$$

Note our use of the compact dot product involving $\mathbf{V}$ and the gradient operator $\boldsymbol{\nabla}$:

$$u\frac{\partial}{\partial x} + \upsilon\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} = \mathbf{V}\cdot\boldsymbol{\nabla} \qquad \text{where} \qquad \boldsymbol{\nabla} = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

The total time derivative—sometimes called the *substantial* or *material* derivative—concept may be applied to any variable, such as the pressure:

$$\frac{dp}{dt} = \frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + \upsilon\frac{\partial p}{\partial y} + w\frac{\partial p}{\partial z} = \frac{\partial p}{\partial t} + (\mathbf{V}\cdot\boldsymbol{\nabla})p$$

$$\frac{D\phi}{Dt} = \frac{\partial\phi}{\partial t} + \frac{\partial\phi}{\partial x}\frac{dx}{dt} + \frac{\partial\phi}{\partial y}\frac{dy}{dt} + \frac{\partial\phi}{\partial z}\frac{dz}{dt}$$

A fluid particle follows the flow, so $\mathrm{d}x/\mathrm{d}t = u$, $\mathrm{d}y/\mathrm{d}t = v$ and $\mathrm{d}z/\mathrm{d}t = w$. Hence the substantive derivative of $\phi$ is given by

$$\frac{D\phi}{Dt} = \frac{\partial\phi}{\partial t} + u\frac{\partial\phi}{\partial x} + v\frac{\partial\phi}{\partial y} + w\frac{\partial\phi}{\partial z} = \frac{\partial\phi}{\partial t} + \mathbf{u}\,.\,\mathrm{grad}\,\phi \qquad (2.7)$$

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As in the case of the mass conservation equation, we are interested in developing equations for rates of change per unit volume. The rate of change of property $\phi$ per unit volume for a fluid particle is given by the product of $D\phi/Dt$ and density $\rho$, hence

$$\rho\frac{D\phi}{Dt} = \rho\left(\frac{\partial\phi}{\partial t} + \mathbf{u} \cdot \operatorname{grad} \phi\right) \tag{2.8}$$

The mass conservation equation contains the mass per unit volume (i.e. the density $\rho$) as the conserved quantity. The sum of the rate of change of density in time and the convective term in the mass conservation equation (2.4) for a fluid element is

$$\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \mathbf{u})$$

The generalisation of these terms for an arbitrary conserved property is

$$\frac{\partial (\rho \phi)}{\partial t} + \mathrm{div}(\rho \phi \mathbf{u}) \tag{2.9}$$

Formula (2.9) expresses the rate of change in time of $\phi$ per unit volume plus the net flow of $\phi$ out of the fluid element per unit volume. It is now rewritten to illustrate its relationship with the substantive derivative of $\phi$:

$$\frac{\partial (\rho \phi)}{\partial t} + \mathrm{div}(\rho \phi \mathbf{u}) = \rho \left[ \frac{\partial \phi}{\partial t} + \mathbf{u} \,.\, \mathrm{grad}\ \phi \right] + \phi \left[ \frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \mathbf{u}) \right]$$

$$= \rho \frac{D\phi}{Dt} \tag{2.10}$$

$$\frac{\partial(\rho\phi)}{\partial t} + \text{div}(\rho\phi\mathbf{u}) = \rho\left[\frac{\partial\phi}{\partial t} + \mathbf{u}\,.\,\text{grad}\,\phi\right] + \phi\left[\frac{\partial\rho}{\partial t} + \text{div}(\rho\mathbf{u})\right]$$

$$= \rho\frac{D\phi}{Dt} \tag{2.10}$$

The term $\phi[(\partial\rho/\partial t) + \text{div}(\rho\mathbf{u})]$ is equal to zero by virtue of mass conservation (2.4). In words, relationship (2.10) states

| | | | | |
|---|---|---|---|---|
| Rate of increase of $\phi$ of fluid element | + | Net rate of flow of $\phi$ out of fluid element | = | Rate of increase of $\phi$ for a fluid particle |

To construct the three components of the momentum equation and the energy equation the relevant entries for $\phi$ and their rates of change per unit volume as defined in (2.8) and (2.10) are given below:

| | | | |
|---|---|---|---|
| $x$-momentum | $u$ | $\rho \frac{Du}{Dt}$ | $\frac{\partial(\rho u)}{\partial t} + \text{div}(\rho u \mathbf{u})$ |
| $y$-momentum | $v$ | $\rho \frac{Dv}{Dt}$ | $\frac{\partial(\rho v)}{\partial t} + \text{div}(\rho v \mathbf{u})$ |
| $z$-momentum | $w$ | $\rho \frac{Dw}{Dt}$ | $\frac{\partial(\rho w)}{\partial t} + \text{div}(\rho w \mathbf{u})$ |
| energy | $E$ | $\rho \frac{DE}{Dt}$ | $\frac{\partial(\rho E)}{\partial t} + \text{div}(\rho E \mathbf{u})$ |

### 2.1.3 Momentum equation in three dimensions

**Newton's second law** states that the rate of change of momentum of a fluid particle equals the sum of the forces on the particle:

| Rate of increase of momentum of fluid particle | = | Sum of forces on fluid particle |
|---|---|---|

The **rates of increase of *x*-, *y*- and *z*-momentum** per unit volume of a fluid particle are given by

$$\rho \frac{Du}{Dt} \qquad \rho \frac{Dv}{Dt} \qquad \rho \frac{Dw}{Dt} \tag{2.11}$$

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- **surface forces**
  - pressure forces
  - viscous forces
  - gravity force
- **body forces**
  - centrifugal force
  - Coriolis force
  - electromagnetic force

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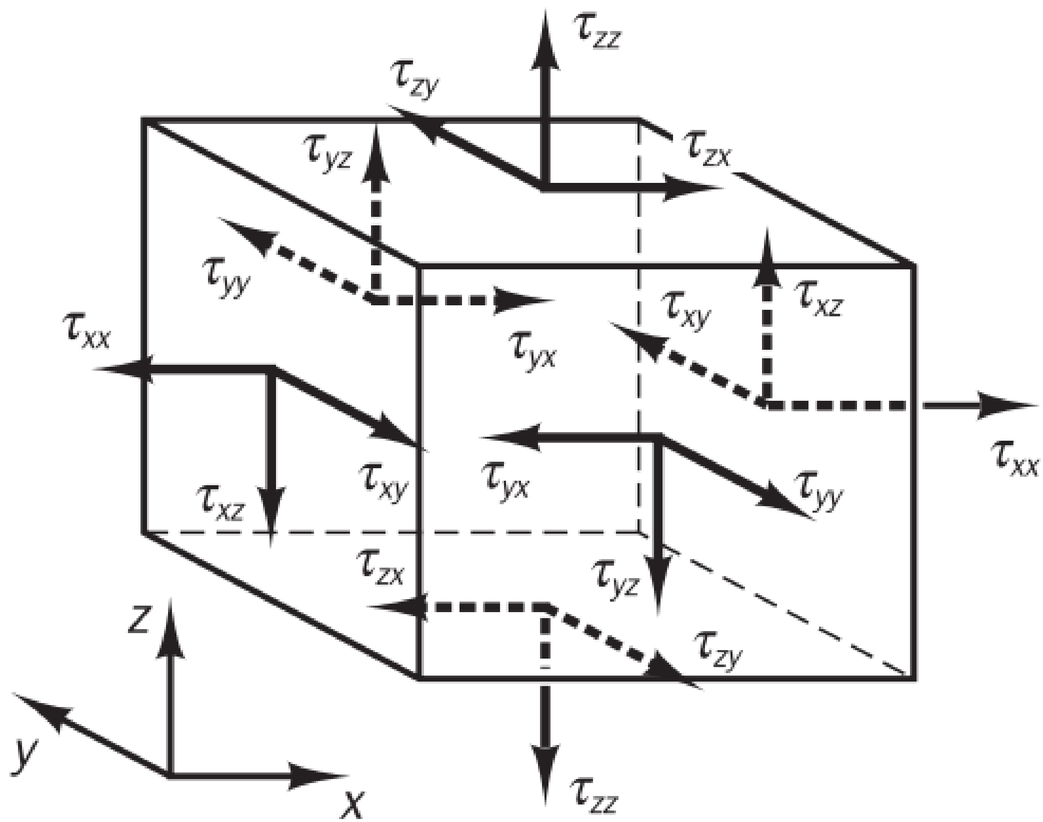


**Figure 2.3** Stress components on three faces of fluid element

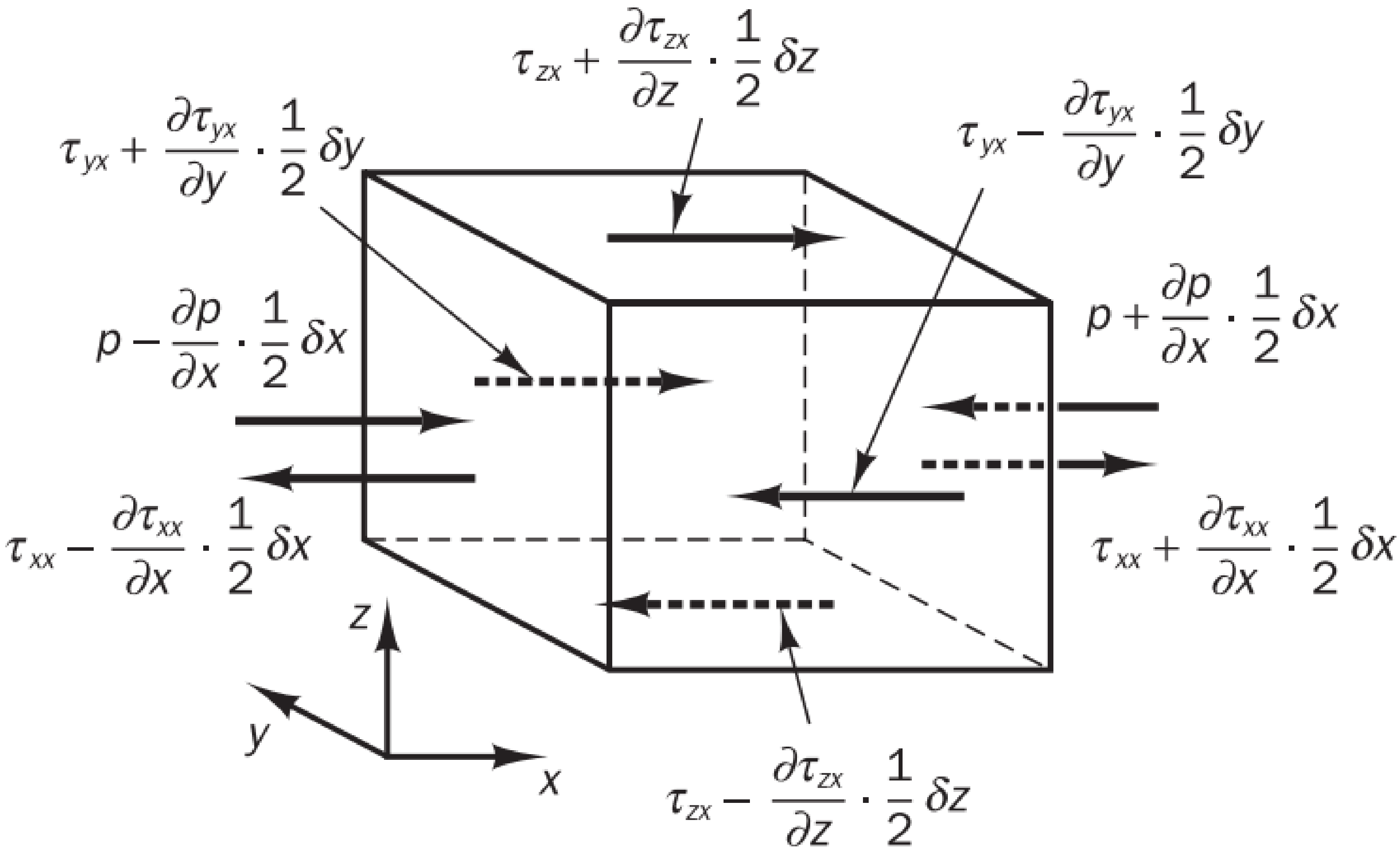


**Figure 2.4** Stress components in the *x*-direction

On the pair of faces $(E, W)$ we have

$$\left[\left(p - \frac{\partial p}{\partial x}\frac{1}{2}\delta x\right) - \left(\tau_{xx} - \frac{\partial \tau_{xx}}{\partial x}\frac{1}{2}\delta x\right)\right]\delta y\,\delta z + \left[-\left(p + \frac{\partial p}{\partial x}\frac{1}{2}\delta x\right)\right.$$

$$\left. + \left(\tau_{xx} + \frac{\partial \tau_{xx}}{\partial x}\frac{1}{2}\delta x\right)\right]\delta y\,\delta z = \left(-\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x}\right)\delta x\,\delta y\,\delta z \qquad (2.12a)$$

The net force in the $x$-direction on the pair of faces $(N, S)$ is

$$-\left(\tau_{yx} - \frac{\partial \tau_{yx}}{\partial y}\frac{1}{2}\delta y\right)\delta x\,\delta z + \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y}\frac{1}{2}\delta y\right)\delta x\,\delta z = \frac{\partial \tau_{yx}}{\partial y}\delta x\,\delta y\,\delta z \qquad (2.12b)$$

Finally the net force in the $x$-direction on faces $T$ and $B$ is given by

$$-\left(\tau_{zx} - \frac{\partial \tau_{zx}}{\partial z}\frac{1}{2}\delta z\right)\delta x\,\delta y + \left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z}\frac{1}{2}\delta z\right)\delta x\,\delta y = \frac{\partial \tau_{zx}}{\partial z}\delta x\,\delta y\,\delta z \qquad (2.12c)$$

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$$\frac{\partial(-p + \tau_{xx})}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}$$

total force in the $x$-direction on the element due to surface stresses (2.13) plus the rate of increase of $x$-momentum due to sources:

$$\rho \frac{Du}{Dt} = \frac{\partial(-p + \tau_{xx})}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + S_{Mx} \tag{2.14a}$$

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It is not too difficult to verify that the **$y$-component of the momentum equation** is given by

$$\boxed{\rho \frac{Dv}{Dt} = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial(-p + \tau_{yy})}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + S_{My}} \tag{2.14b}$$

and the **$z$-component of the momentum equation** by

$$\boxed{\rho \frac{Dw}{Dt} = \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial(-p + \tau_{zz})}{\partial z} + S_{Mz}} \tag{2.14c}$$

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The effects of surface stresses are accounted for explicitly; the source terms $S_{Mx}$, $S_{My}$ and $S_{Mz}$ in (2.14a–c) include contributions due to body forces only. For example, the body force due to gravity would be modelled by $S_{Mx} = 0$, $S_{My} = 0$ and $S_{Mz} = -\rho g$.

## 2.1.4 Energy equation in three dimensions

The energy equation is derived from the **first law of thermodynamics**, which states that the rate of change of energy of a fluid particle is equal to the rate of heat addition to the fluid particle plus the rate of work done on the particle:

| | | | | |
|---|---|---|---|---|
| Rate of increase of energy of fluid particle | = | Net rate of heat added to fluid particle | + | Net rate of work done on fluid particle |

As before, we will be deriving an equation for the **rate of increase of energy** of a fluid particle per unit volume, which is given by

$$\rho \frac{DE}{Dt} \tag{2.15}$$

### *Work done by surface forces*

The **rate of work done** on the fluid particle in the element by a **surface force** is equal to the product of the force and velocity component in the direction of the force. For example, the forces given by (2.12a–c) all act in the $x$-direction. The work done by these forces is given by

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$$\left[\left(pu - \frac{\partial(pu)}{\partial x}\frac{1}{2}\delta x\right) - \left(\tau_{xx}u - \frac{\partial(\tau_{xx}u)}{\partial x}\frac{1}{2}\delta x\right)\right.$$

$$\left. - \left(pu + \frac{\partial(pu)}{\partial x}\frac{1}{2}\delta x\right) + \left(\tau_{xx}u + \frac{\partial(\tau_{xx}u)}{\partial x}\frac{1}{2}\delta x\right)\right]\delta y\delta z$$

$$+ \left[-\left(\tau_{yx}u - \frac{\partial(\tau_{yx}u)}{\partial y}\frac{1}{2}\delta y\right) + \left(\tau_{yx}u + \frac{\partial(\tau_{yx}u)}{\partial y}\frac{1}{2}\delta y\right)\right]\delta x\delta z$$

$$+ \left[-\left(\tau_{zx}u - \frac{\partial(\tau_{zx}u)}{\partial z}\frac{1}{2}\delta z\right) + \left(\tau_{zx}u + \frac{\partial(\tau_{zx}u)}{\partial z}\frac{1}{2}\delta z\right)\right]\delta x\delta y$$

The net rate of work done by these surface forces acting in the $x$-direction is given by

$$\left[\frac{\partial(u(-p+\tau_{xx}))}{\partial x}+\frac{\partial(u\tau_{yx})}{\partial y}+\frac{\partial(u\tau_{zx})}{\partial z}\right]\delta x\,\delta y\,\delta z \qquad (2.16a)$$

Surface stress components in the $y$- and $z$-direction also do work on the fluid particle. A repetition of the above process gives the additional rates of work done on the fluid particle due to the work done by these surface forces:

$$\left[\frac{\partial(v\tau_{xy})}{\partial x}+\frac{\partial(v(-p+\tau_{yy}))}{\partial y}+\frac{\partial(v\tau_{zy})}{\partial z}\right]\delta x\,\delta y\,\delta z \qquad (2.16b)$$

and

$$\left[\frac{\partial(w\tau_{xz})}{\partial x}+\frac{\partial(w\tau_{yz})}{\partial y}+\frac{\partial(w(-p+\tau_{zz}))}{\partial z}\right]\delta x\,\delta y\,\delta z \qquad (2.16c)$$

The total rate of work done per unit volume on the fluid particle by all the surface forces is given by the sum of (2.16a–c) divided by the volume $\delta x \delta y \delta z$. The terms containing pressure can be collected together and written more compactly in vector form

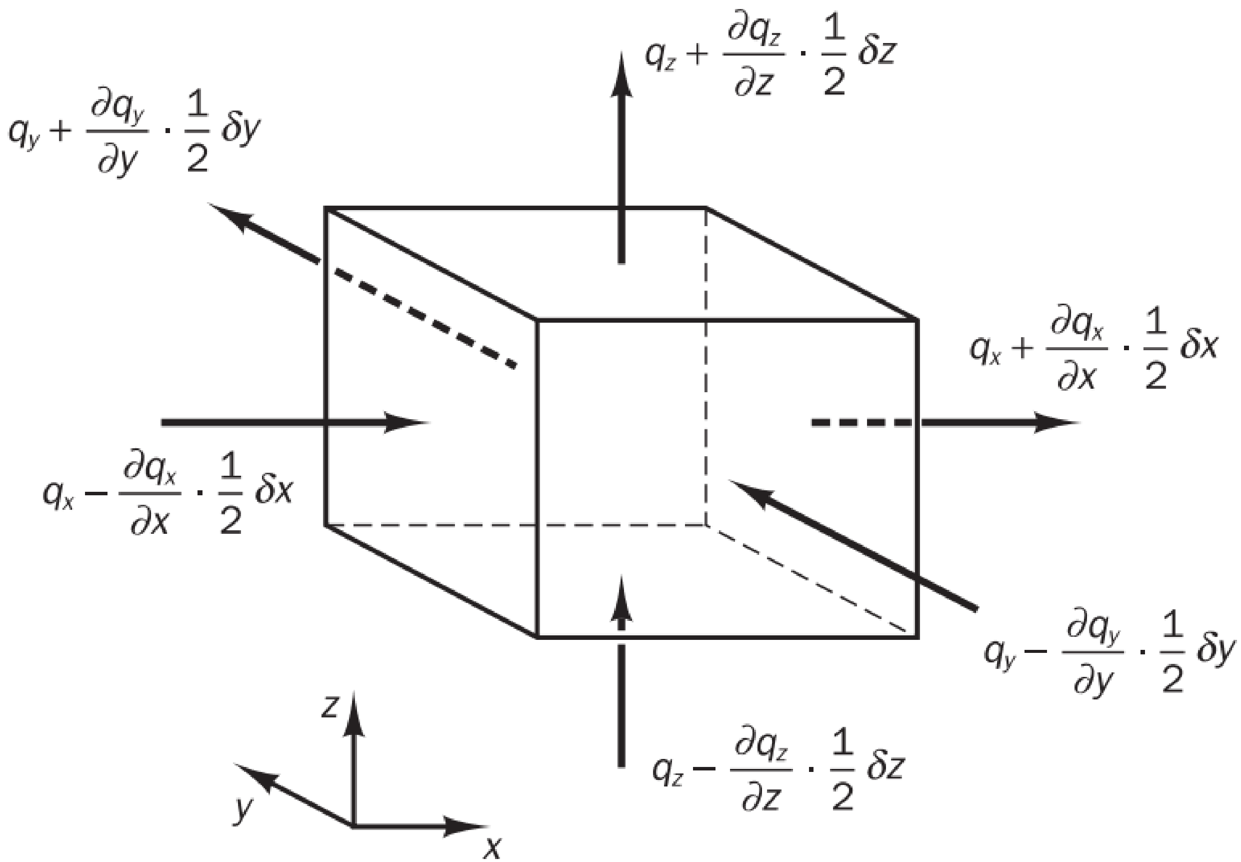
$$-\frac{\partial(up)}{\partial x} - \frac{\partial(vp)}{\partial y} - \frac{\partial(wp)}{\partial z} = -\text{div}(p\mathbf{u})$$

This yields the following **total rate of work done on the fluid particle by surface stresses**:

$$[-\text{div}(p\mathbf{u})] + \left[ \frac{\partial(u\tau_{xx})}{\partial x} + \frac{\partial(u\tau_{yx})}{\partial y} + \frac{\partial(u\tau_{zx})}{\partial z} + \frac{\partial(v\tau_{xy})}{\partial x} + \frac{\partial(v\tau_{yy})}{\partial y} \right.$$
$$\left. + \frac{\partial(v\tau_{zy})}{\partial z} + \frac{\partial(w\tau_{xz})}{\partial x} + \frac{\partial(w\tau_{yz})}{\partial y} + \frac{\partial(w\tau_{zz})}{\partial z} \right] \tag{2.17}$$

### *Energy flux due to heat conduction*

The heat flux vector **q** has three components: $q_x$, $q_y$ and $q_z$ (Figure 2.5).



The **net rate of heat transfer to the fluid particle** due to heat flow in the $x$-direction is given by the difference between the rate of heat input across face $W$ and the rate of heat loss across face $E$:

$$\left[\left(q_x - \frac{\partial q_x}{\partial x}\frac{1}{2}\delta x\right) - \left(q_x + \frac{\partial q_x}{\partial x}\frac{1}{2}\delta x\right)\right]\delta y\,\delta z = -\frac{\partial q_x}{\partial x}\delta x\,\delta y\,\delta z \quad (2.18a)$$

Similarly, the net rates of heat transfer to the fluid due to heat flows in the $y$- and $z$-direction are

$$-\frac{\partial q_y}{\partial y}\delta x\,\delta y\,\delta z \quad \text{and} \quad -\frac{\partial q_z}{\partial z}\delta x\,\delta y\,\delta z \quad (2.18b\text{–}c)$$

The total rate of heat added to the fluid particle per unit volume due to heat flow across its boundaries is the sum of (2.18a–c) divided by the volume $\delta x\,\delta y\,\delta z$:

$$-\frac{\partial q_x}{\partial x} - \frac{\partial q_y}{\partial y} - \frac{\partial q_z}{\partial z} = -\text{div}\,\mathbf{q} \quad (2.19)$$

Fourier's law of heat conduction relates the heat flux to the local temperature gradient. So

$$q_x = -k\frac{\partial T}{\partial x} \quad q_y = -k\frac{\partial T}{\partial y} \quad q_z = -k\frac{\partial T}{\partial z}$$

This can be written in vector form as follows:

$$\mathbf{q} = -k \text{ grad } T \tag{2.20}$$

Combining (2.19) and (2.20) yields the final form of the **rate of heat addition to the fluid particle due to heat conduction** across element boundaries:

$$-\text{div } \mathbf{q} = \text{div}(k \text{ grad } T) \tag{2.21}$$

### Energy equation

Thus far we have not defined the specific energy $E$ of a fluid. Often the energy of a fluid is defined as the sum of internal (thermal) energy $i$, kinetic energy $\frac{1}{2}(u^2 + v^2 + w^2)$ and gravitational potential energy. This definition

$$\rho \frac{DE}{Dt} = -\mathrm{div}(p\mathbf{u}) + \left[ \frac{\partial(u\tau_{xx})}{\partial x} + \frac{\partial(u\tau_{yx})}{\partial y} + \frac{\partial(u\tau_{zx})}{\partial z} + \frac{\partial(v\tau_{xy})}{\partial x} + \frac{\partial(v\tau_{yy})}{\partial y} + \frac{\partial(v\tau_{zy})}{\partial z} + \frac{\partial(w\tau_{xz})}{\partial x} + \frac{\partial(w\tau_{yz})}{\partial y} + \frac{\partial(w\tau_{zz})}{\partial z} \right] + \mathrm{div}(k \ \mathrm{grad} \ T) + S_E \tag{2.22}$$

$$E = i + \frac{1}{2}(u^2 + v^2 + w^2).$$

$$E = i + \tfrac{1}{2}(u^2 + v^2 + w^2).$$

conservation equation for the kinetic energy:

$$\boxed{\begin{aligned} \rho \frac{D[\frac{1}{2}(u^2 + v^2 + w^2)]}{Dt} = -\mathbf{u} \,.\, \text{grad}\, p &+ u\left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right) \\ &+ v\left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right) \\ &+ w\left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}\right) + \mathbf{u} \,.\, \mathbf{S}_M \end{aligned}} \tag{2.23}$$

$$E = i + \frac{1}{2}(u^2 + v^2 + w^2).$$

Subtracting (2.23) from (2.22) and defining a new source term as $S_i = S_E - \mathbf{u} \cdot \mathbf{S}_M$ yields the internal energy equation

$$\rho\frac{Di}{Dt} = -p \operatorname{div} \mathbf{u} + \operatorname{div}(k \operatorname{grad} T) + \tau_{xx}\frac{\partial u}{\partial x} + \tau_{yx}\frac{\partial u}{\partial y} + \tau_{zx}\frac{\partial u}{\partial z} + \tau_{xy}\frac{\partial v}{\partial x} + \tau_{yy}\frac{\partial v}{\partial y} + \tau_{zy}\frac{\partial v}{\partial z} + \tau_{xz}\frac{\partial w}{\partial x} + \tau_{yz}\frac{\partial w}{\partial y} + \tau_{zz}\frac{\partial w}{\partial z} + S_i \tag{2.24}$$

For the special case of an incompressible fluid we have $i = cT$, where $c$ is the specific heat and div $\mathbf{u} = 0$. This allows us to recast (2.24) into a temperature equation

$$\boxed{\begin{aligned}\rho c\frac{DT}{Dt} = \operatorname{div}(k \operatorname{grad} T) &+ \tau_{xx}\frac{\partial u}{\partial x} + \tau_{yx}\frac{\partial u}{\partial y} + \tau_{zx}\frac{\partial u}{\partial z} + \tau_{xy}\frac{\partial v}{\partial x} \\ &+ \tau_{yy}\frac{\partial v}{\partial y} + \tau_{zy}\frac{\partial v}{\partial z} + \tau_{xz}\frac{\partial w}{\partial x} + \tau_{yz}\frac{\partial w}{\partial y} + \tau_{zz}\frac{\partial w}{\partial z} + S_i\end{aligned}} \qquad (2.25)$$

For compressible flows equation (2.22) is often rearranged to give an equation for the **enthalpy**. The specific enthalpy $h$ and the specific total enthalpy $h_0$ of a fluid are defined as

$$h = i + p/\rho \quad \text{and} \quad h_0 = h + \tfrac{1}{2}(u^2 + v^2 + w^2)$$

Combining these two definitions with the one for specific energy $E$ we get

$$h_0 = i + p/\rho + \tfrac{1}{2}(u^2 + v^2 + w^2) = E + p/\rho \tag{2.26}$$

Substitution of (2.26) into (2.22) and some rearrangement yields the **(total) enthalpy equation**

$$\begin{aligned}
\frac{\partial(\rho h_0)}{\partial t} + \operatorname{div}(\rho h_0 \mathbf{u}) = \operatorname{div}(k \operatorname{grad} T) + \frac{\partial p}{\partial t} \\
+ \left[ \frac{\partial(u\tau_{xx})}{\partial x} + \frac{\partial(u\tau_{yx})}{\partial y} + \frac{\partial(u\tau_{zx})}{\partial z} \right. \\
+ \frac{\partial(v\tau_{xy})}{\partial x} + \frac{\partial(v\tau_{yy})}{\partial y} + \frac{\partial(v\tau_{zy})}{\partial z} \\
\left. + \frac{\partial(w\tau_{xz})}{\partial x} + \frac{\partial(w\tau_{yz})}{\partial y} + \frac{\partial(w\tau_{zz})}{\partial z} \right] + S_h
\end{aligned} \tag{2.27}$$

It should be stressed that equations (2.24), (2.25) and (2.27) are *not* new (extra) conservation laws but merely alternative forms of the energy equation (2.22).

## 2.2 Equations of state

We can describe the state of a substance in thermodynamic equilibrium by means of just two state variables. **Equations of state** relate the other variables to the two state variables. If we use $\rho$ and $T$ as state variables we have state equations for pressure $p$ and specific internal energy $i$:

$$p = p(\rho, T) \quad \text{and} \quad i = i(\rho, T) \tag{2.28}$$

For a **perfect gas** the following, well-known, equations of state are useful:

$$p = \rho RT \quad \text{and} \quad i = C_v T \tag{2.29}$$

## 2.2 Equations of state

Liquids and gases flowing at low speeds behave as **incompressible fluids**. Without density variations there is no linkage between the energy equation and the mass conservation and momentum equations. The flow field can often be solved by considering mass conservation and momentum equations only. The energy equation only needs to be solved alongside the others if the problem involves heat transfer.

## 2.3 Navier–Stokes equations for a Newtonian fluid

The volumetric deformation is given by

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \operatorname{div} \mathbf{u} \tag{2.30c}$$

$$\tau_{xx} = 2\mu \frac{\partial u}{\partial x} + \lambda \operatorname{div} \mathbf{u} \quad \tau_{yy} = 2\mu \frac{\partial v}{\partial y} + \lambda \operatorname{div} \mathbf{u} \quad \tau_{zz} = 2\mu \frac{\partial w}{\partial z} + \lambda \operatorname{div} \mathbf{u}$$

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \quad \tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \tag{2.31}$$

$$\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left[2\mu \frac{\partial u}{\partial x} + \lambda \operatorname{div} \mathbf{u}\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right] + S_{Mx} \tag{2.32a}$$

$$\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left[2\mu \frac{\partial u}{\partial x} + \lambda \operatorname{div} \mathbf{u}\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right] + S_{Mx} \tag{2.32a}$$

$$\rho \frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] + \frac{\partial}{\partial y}\left[2\mu \frac{\partial v}{\partial y} + \lambda \operatorname{div} \mathbf{u}\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right] + S_{My} \tag{2.32b}$$

$$\rho \frac{Dw}{Dt} = -\frac{\partial p}{\partial z} + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right] + \frac{\partial}{\partial z}\left[2\mu \frac{\partial w}{\partial z} + \lambda \operatorname{div} \mathbf{u}\right] + S_{Mz} \tag{2.32c}$$

Often it is useful to rearrange the viscous stress terms as follows:

$$\frac{\partial}{\partial x}\left[2\mu\frac{\partial u}{\partial x}+\lambda\,\mathrm{div}\,\mathbf{u}\right]+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right]$$

$$=\frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial z}\left(\mu\frac{\partial u}{\partial z}\right)$$

$$+\left[\frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial z}\left(\mu\frac{\partial w}{\partial x}\right)+\frac{\partial}{\partial x}(\lambda\,\mathrm{div}\,\mathbf{u})\right]$$

$$=\mathrm{div}(\mu\,\mathrm{grad}\,u)+[s_{Mx}]$$

The viscous stresses in the *y*- and *z*-component equations can be recast in a similar manner. We clearly intend to simplify the momentum equations by 'hiding' the bracketed smaller contributions to the viscous stress terms in the momentum source. Defining a new source by

$$S_M = S_M + [s_M] \tag{2.33}$$

the **Navier–Stokes equations** can be written in the most useful form for the development of the finite volume method:

$$\boxed{\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \text{div}(\mu \text{ grad } u) + S_{Mx}} \tag{2.34a}$$

$$\boxed{\rho \frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + \text{div}(\mu \text{ grad } v) + S_{My}} \tag{2.34b}$$

$$\boxed{\rho \frac{Dw}{Dt} = -\frac{\partial p}{\partial z} + \text{div}(\mu \text{ grad } w) + S_{Mz}} \tag{2.34c}$$

If we use the Newtonian model for viscous stresses in the internal energy equation (2.24) we obtain after some rearrangement

$$\boxed{\rho \frac{Di}{Dt} = -p \operatorname{div} \mathbf{u} + \operatorname{div}(k \operatorname{grad} T) + \Phi + S_i} \tag{2.35}$$

All the effects due to viscous stresses in this internal energy equation are described by the dissipation function $\Phi$, which, after considerable algebra, can be shown to be equal to

$$\Phi = \mu \left\{ 2\left[ \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2 \right] + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)^2 \right\} + \lambda(\operatorname{div} \mathbf{u})^2 \tag{2.36}$$

## 2.4 Conservative form of the governing equations of fluid flow

**Table 2.1** Governing equations of the flow of a compressible Newtonian fluid

| | | |
|---|---|---|
| Continuity | $\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{u}) = 0$ | (2.4) |
| *x*-momentum | $\frac{\partial(\rho u)}{\partial t} + \operatorname{div}(\rho u \mathbf{u}) = -\frac{\partial p}{\partial x} + \operatorname{div}(\mu \operatorname{grad} u) + S_{Mx}$ | (2.37a) |
| *y*-momentum | $\frac{\partial(\rho v)}{\partial t} + \operatorname{div}(\rho v \mathbf{u}) = -\frac{\partial p}{\partial y} + \operatorname{div}(\mu \operatorname{grad} v) + S_{My}$ | (2.37b) |
| *z*-momentum | $\frac{\partial(\rho w)}{\partial t} + \operatorname{div}(\rho w \mathbf{u}) = -\frac{\partial p}{\partial z} + \operatorname{div}(\mu \operatorname{grad} w) + S_{Mz}$ | (2.37c) |
| Energy | $\frac{\partial(\rho i)}{\partial t} + \operatorname{div}(\rho i \mathbf{u}) = -p \operatorname{div} \mathbf{u} + \operatorname{div}(k \operatorname{grad} T) + \Phi + S_i$ | (2.38) |
| Equations of state | $p = p(\rho, T)$ and $i = i(\rho, T)$ | (2.28) |
| | e.g. perfect gas $p = \rho RT$ and $i = C_v T$ | (2.29) |

## 2.5 Differential and integral forms of the general transport equations

It is clear from Table 2.1 that there are significant commonalities between the various equations. If we introduce a general variable $\phi$ the conservative form of all fluid flow equations, including equations for scalar quantities such as temperature and pollutant concentration etc., can usefully be written in the following form:

$$\frac{\partial(\rho\phi)}{\partial t} + \operatorname{div}(\rho\phi\mathbf{u}) = \operatorname{div}(\Gamma \operatorname{grad} \phi) + S_\phi \tag{2.39}$$

## 2.5 Differential and integral forms of the general transport equations

It is clear from Table 2.1 that there are significant commonalities between the various equations. If we introduce a general variable $\phi$ the conservative form of all fluid flow equations, including equations for scalar quantities such as temperature and pollutant concentration etc., can usefully be written in the following form:

$$\frac{\partial(\rho\phi)}{\partial t} + \operatorname{div}(\rho\phi\mathbf{u}) = \operatorname{div}(\Gamma \operatorname{grad} \phi) + S_\phi \tag{2.39}$$

| Rate of increase of $\phi$ of fluid element | + | Net rate of flow of $\phi$ out of fluid element | = | Rate of increase of $\phi$ due to diffusion | + | Rate of increase of $\phi$ due to sources |
|---|---|---|---|---|---|---|

$$\frac{\partial(\rho\phi)}{\partial t} + \operatorname{div}(\rho\phi\mathbf{u}) = \operatorname{div}(\Gamma \operatorname{grad} \phi) + S_\phi$$

1 fluid

| | | |
|---|---|---|
| Continuity | $\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho\mathbf{u}) = 0$ | (2.4) |
| *x*-momentum | $\frac{\partial(\rho u)}{\partial t} + \operatorname{div}(\rho u\mathbf{u}) = -\frac{\partial p}{\partial x} + \operatorname{div}(\mu \operatorname{grad} u) + S_{Mx}$ | (2.37a) |
| *y*-momentum | $\frac{\partial(\rho v)}{\partial t} + \operatorname{div}(\rho v\mathbf{u}) = -\frac{\partial p}{\partial y} + \operatorname{div}(\mu \operatorname{grad} v) + S_{My}$ | (2.37b) |
| *z*-momentum | $\frac{\partial(\rho w)}{\partial t} + \operatorname{div}(\rho w\mathbf{u}) = -\frac{\partial p}{\partial z} + \operatorname{div}(\mu \operatorname{grad} w) + S_{Mz}$ | (2.37c) |
| Energy | $\frac{\partial(\rho i)}{\partial t} + \operatorname{div}(\rho i\mathbf{u}) = -p \operatorname{div} \mathbf{u} + \operatorname{div}(k \operatorname{grad} T) + \Phi + S_i$ | (2.38) |
| Equations of state | $p = p(\rho, T)$ and $i = i(\rho, T)$ | (2.28) |
| | e.g. perfect gas $p = \rho RT$ and $i = C_v T$ | (2.29) |

$$\frac{\partial(\rho\phi)}{\partial t} + \operatorname{div}(\rho\phi\mathbf{u}) = \operatorname{div}(\Gamma \operatorname{grad} \phi) + S_\phi$$

$$\frac{\partial(\rho\phi)}{\partial t} + \operatorname{div}(\rho\phi\mathbf{u}) = \operatorname{div}(\Gamma \operatorname{grad} \phi) + S_\phi$$

$$\int_{CV} \frac{\partial(\rho\phi)}{\partial t} dV + \int_{CV} \operatorname{div}(\rho\phi\mathbf{u}) dV = \int_{CV} \operatorname{div}(\Gamma \operatorname{grad} \phi) dV + \int_{CV} S_\phi dV \qquad (2.40)$$

$$\frac{\partial(\rho\phi)}{\partial t} + \operatorname{div}(\rho\phi\mathbf{u}) = \operatorname{div}(\Gamma \operatorname{grad} \phi) + S_\phi$$

$$\int_{CV} \frac{\partial(\rho\phi)}{\partial t} \mathrm{d}V + \int_{CV} \operatorname{div}(\rho\phi\mathbf{u}) \mathrm{d}V = \int_{CV} \operatorname{div}(\Gamma \operatorname{grad} \phi) \mathrm{d}V + \int_{CV} S_\phi \mathrm{d}V \qquad (2.40)$$

$$\int_{CV} \operatorname{div}(\mathbf{a}) \mathrm{d}V = \int_{A} \mathbf{n} \,.\, \mathbf{a} \mathrm{d}A \qquad (2.41)$$

$$\frac{\partial(\rho\phi)}{\partial t} + \mathrm{div}(\rho\phi\mathbf{u}) = \mathrm{div}(\Gamma\ \mathrm{grad}\ \phi) + S_\phi$$

$$\int_{CV} \frac{\partial(\rho\phi)}{\partial t}\mathrm{d}V + \int_{CV} \mathrm{div}(\rho\phi\mathbf{u})\mathrm{d}V = \int_{CV} \mathrm{div}(\Gamma\ \mathrm{grad}\ \phi)\mathrm{d}V + \int_{CV} S_\phi \mathrm{d}V \qquad (2.40)$$

$$\int_{CV} \mathrm{div}(\mathbf{a})\mathrm{d}V = \int_{A} \mathbf{n}\ .\ \mathbf{a}\mathrm{d}A \qquad (2.41)$$

$$\frac{\partial}{\partial t}\left(\int_{CV} \rho\phi dV\right) + \int_{A} \mathbf{n}\ .\ (\rho\phi\mathbf{u})\mathrm{d}A = \int_{A} \mathbf{n}\ .\ (\Gamma\ \mathrm{grad}\ \phi)\mathrm{d}A + \int_{CV} S_\phi \mathrm{d}V \qquad (2.42)$$

$$\frac{\partial}{\partial t}\left(\int_{CV} \rho\phi dV\right) + \int_A \mathbf{n} \cdot (\rho\phi\mathbf{u}) dA = \int_A \mathbf{n} \cdot (\Gamma \operatorname{grad} \phi) dA + \int_{CV} S_\phi dV \quad (2.42)$$

| | | | | | | |
|---|---|---|---|---|---|---|
| Rate of increase of $\phi$ inside the control volume | + | Net rate of decrease of $\phi$ due to convection across the control volume boundaries | = | Net rate of increase of $\phi$ due to diffusion across the control volume boundaries | + | Net rate of creation of $\phi$ inside the control volume |

$$\frac{\partial}{\partial t}\left(\int_{CV} \rho\phi dV\right) + \int_A \mathbf{n}\,.\,(\rho\phi\mathbf{u})\mathrm{d}A = \int_A \mathbf{n}\,.\,(\Gamma\,\mathrm{grad}\,\phi)\mathrm{d}A + \int_{CV} S_\phi \mathrm{d}V \quad (2.42)$$

In steady state problems the rate of change term of (2.42) is equal to zero. This leads to the integrated form of the steady transport equation:

$$\int_A \mathbf{n}\,.\,(\rho\phi\mathbf{u})\mathrm{d}A = \int_A \mathbf{n}\,.\,(\Gamma\,\mathrm{grad}\,\phi)\mathrm{d}A + \int_{CV} S_\phi \mathrm{d}V \qquad (2.43)$$

$$\frac{\partial}{\partial t}\left(\int_{CV} \rho\phi dV\right) + \int_A \mathbf{n} \,.\, (\rho\phi\mathbf{u})\mathrm{d}A = \int_A \mathbf{n} \,.\, (\Gamma \,\mathrm{grad}\, \phi)\mathrm{d}A + \int_{CV} S_\phi \mathrm{d}V \quad (2.42)$$

In time-dependent problems it is also necessary to integrate with respect to time $t$ over a small interval $\Delta t$ from, say, $t$ until $t + \Delta t$. This yields the most general integrated form of the transport equation:

$$\int_{\Delta t} \frac{\partial}{\partial t}\left(\int_{CV} \rho\phi \mathrm{d}V\right)\mathrm{d}t + \int_{\Delta t}\int_A \mathbf{n} \,.\, (\rho\phi\mathbf{u})\mathrm{d}A\mathrm{d}t$$

$$= \int_{\Delta t}\int_A \mathbf{n} \,.\, (\Gamma \,\mathrm{grad}\, \phi)\mathrm{d}A\mathrm{d}t + \int_{\Delta t}\int_{CV} S_\phi \mathrm{d}V\mathrm{d}t \qquad (2.44)$$