Chapter six Solution algorithms for pressure–velocity coupling in steady flows

Navier-Stokes Equations

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \dfrac{\partial (\rho v_i)}{\partial x_i} = 0 \\ \dfrac{\partial \rho v_i}{\partial t} + \dfrac{\partial (\rho v_j v_i)}{\partial x_j} = \dfrac{\partial \tau_{ij}}{\partial x_j} - \dfrac{\partial p}{\partial x_i} \\ \dfrac{\partial (\rho E)}{\partial t} + \dfrac{\partial (\rho v_j E)}{\partial x_j} = \dfrac{\partial}{\partial x_j}\left(k \dfrac{\partial T}{\partial x_j}\right) + \dfrac{\partial}{\partial x_j}(\tau_{ij} v_i) \end{cases}$$

Conservation of mass

Conservation of momentum

Conservation of energy

Newtonian fluid $\tau_{ij} = \mu \left[\left(\dfrac{\partial v_i}{\partial x_j} + \dfrac{\partial v_j}{\partial x_i} \right) - \dfrac{2}{3} \dfrac{\partial v_k}{\partial x_k} \right]$

Navier Stokes Equations

Continuity equation

$$\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}=0$$

Momentum x

$$\rho(\frac{\partial v_x}{\partial \tau}+v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}+v_z\frac{\partial v_x}{\partial z})=-\frac{\partial p}{\partial x}+\mu\frac{\partial^2 v_x}{\partial x^2}+\mu\frac{\partial^2 v_x}{\partial y^2}+\mu\frac{\partial^2 v_x}{\partial z^2}+S_{Mx}$$

Momentum y

$$\rho(\frac{\partial v_y}{\partial \tau}+v_x\frac{\partial v_y}{\partial x}+v_y\frac{\partial v_y}{\partial y}+v_z\frac{\partial v_y}{\partial z})=-\frac{\partial p}{\partial y}+\mu\frac{\partial^2 v_y}{\partial x^2}+\mu\frac{\partial^2 v_y}{\partial y^2}+\mu\frac{\partial^2 v_y}{\partial z^2}+S_{My}-\rho_\infty g\beta(T-T_\infty)$$

Momentum z

$$\rho(\frac{\partial v_z}{\partial \tau}+v_x\frac{\partial v_z}{\partial x}+v_y\frac{\partial v_z}{\partial y}+v_z\frac{\partial v_z}{\partial z})=-\frac{\partial p}{\partial z}+\mu\frac{\partial^2 v_z}{\partial x^2}+\mu\frac{\partial^2 v_z}{\partial y^2}+\mu\frac{\partial^2 v_z}{\partial z^2}+S_{Mz}$$

Navier Stokes Equations

Continuity equation

$$\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}=0$$

This velocities that constitute advection coefficients: F=ρV

Momentum x

$$\rho\left(\frac{\partial v_x}{\partial \tau}+v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}+v_z\frac{\partial v_x}{\partial z}\right)=-\frac{\partial p}{\partial x}+\mu\frac{\partial^2 v_x}{\partial x^2}+\mu\frac{\partial^2 v_x}{\partial y^2}+\mu\frac{\partial^2 v_x}{\partial z^2}+S_{Mx}$$

Momentum y

$$\rho\left(\frac{\partial v_y}{\partial \tau}+v_x\frac{\partial v_y}{\partial x}+v_y\frac{\partial v_y}{\partial y}+v_z\frac{\partial v_y}{\partial z}\right)=-\frac{\partial p}{\partial y}+\mu\frac{\partial^2 v_y}{\partial x^2}+\mu\frac{\partial^2 v_y}{\partial y^2}+\mu\frac{\partial^2 v_y}{\partial z^2}+S_{My}-\rho_\infty g\beta(T-T_\infty)$$

Momentum z

$$\rho\left(\frac{\partial v_z}{\partial \tau}+v_x\frac{\partial v_z}{\partial x}+v_y\frac{\partial v_z}{\partial y}+v_z\frac{\partial v_z}{\partial z}\right)=-\frac{\partial p}{\partial z}+\mu\frac{\partial^2 v_z}{\partial x^2}+\mu\frac{\partial^2 v_z}{\partial y^2}+\mu\frac{\partial^2 v_z}{\partial z^2}+S_{Mz}$$

Pressure is in momentum equations which already has one unknown

Navier Stokes Equations

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Momentum x

$$\rho\left(\frac{\partial v_x}{\partial \tau}+v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}+v_z\frac{\partial v_x}{\partial z}\right)=-\frac{\partial p}{\partial x}+\mu\frac{\partial^2 v_x}{\partial x^2}+\mu\frac{\partial^2 v_x}{\partial y^2}+\mu\frac{\partial^2 v_x}{\partial z^2}+S_{Mx}$$

Momentum y

$$\rho\left(\frac{\partial v_y}{\partial \tau}+v_x\frac{\partial v_y}{\partial x}+v_y\frac{\partial v_y}{\partial y}+v_z\frac{\partial v_y}{\partial z}\right)=-\frac{\partial p}{\partial y}+\mu\frac{\partial^2 v_y}{\partial x^2}+\mu\frac{\partial^2 v_y}{\partial y^2}+\mu\frac{\partial^2 v_y}{\partial z^2}+S_{My}-\rho_\infty g\beta(T-T_\infty)$$

Momentum z

$$\rho\left(\frac{\partial v_z}{\partial \tau}+v_x\frac{\partial v_z}{\partial x}+v_y\frac{\partial v_z}{\partial y}+v_z\frac{\partial v_z}{\partial z}\right)=-\frac{\partial p}{\partial z}+\mu\frac{\partial^2 v_z}{\partial x^2}+\mu\frac{\partial^2 v_z}{\partial y^2}+\mu\frac{\partial^2 v_z}{\partial z^2}+S_{Mz}$$

Pressure is in momentum equations which already has one unknown

In order to use linear equation solver we need to solve two problems:

1) **find velocities that constitute in advection coefficients**
2) **link pressure field with continuity equation**

Pressure and velocities in NS equations

How to find velocities that constitute in advection coefficients?

$$\rho\left(\frac{\partial v_x}{\partial \tau} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z}\right) = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 v_x}{\partial x^2} + \mu \frac{\partial^2 v_x}{\partial y^2} + \mu \frac{\partial^2 v_x}{\partial z^2} + S_{Mx}$$

$$a_P V_{x,P} + a_E V_{x,E} + a_W V_{x,W} + a_S V_{x,S} + a_N V_{x,N} + a_H V_{x,H} + a_L V_{x,P_L} = f$$

$$a_P = 6\frac{\mu}{\Delta x^2} - \rho \frac{V_x + V_y + V_z}{\Delta x}$$

$$a_E = -\frac{\mu}{\Delta x^2} + \rho \frac{V_x}{\Delta x}, \quad a_W = -\frac{\mu}{\Delta x^2}$$

....

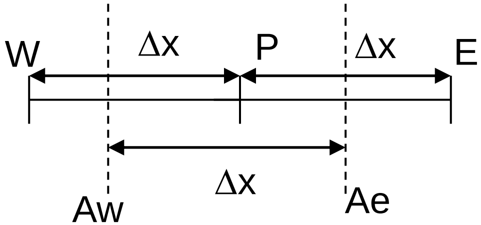
For the first step use Initial guess
And for next iterative steps use
the values from previous iteration

Pressure and velocities in NS equations

How to link pressure field with continuity equation?

SIMPLE (**Semi-Implicit** Method for Pressure-Linked Equations) algorithm

$$\rho\left(\frac{\partial v_x}{\partial \tau} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z}\right) = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 v_x}{\partial x^2} + \mu \frac{\partial^2 v_x}{\partial y^2} + \mu \frac{\partial^2 v_x}{\partial z^2} + S_{Mx}$$



$Aw=Ae=A_{side}$

$$-\frac{\partial p}{\partial x} \approx \frac{P_w - P_e}{\Delta x} = \frac{(P_W + P_P)/2 - (P_P + P_E)/2}{\Delta x} \approx \frac{(P_W - P_E)/2}{\Delta x}$$

$$a_P V_{xP} + a_E V_{xE} + a_W V_{xW} + a_S V_{xS} + a_N V_{xN} + a_H V_{xH} + a_L V_{xL} = f - \frac{(P_W - P_E)/2}{\Delta x} A_{side}$$

We have two additional equations for y and x directions

The momentum equations can be solved only when the pressure field is given or is somehow estimated. Use * for estimated pressure and the corresponding velocities

2D Governing Equation of Laminar Steady Flow (Navier-Stokes)

- x-Momentum Equation:

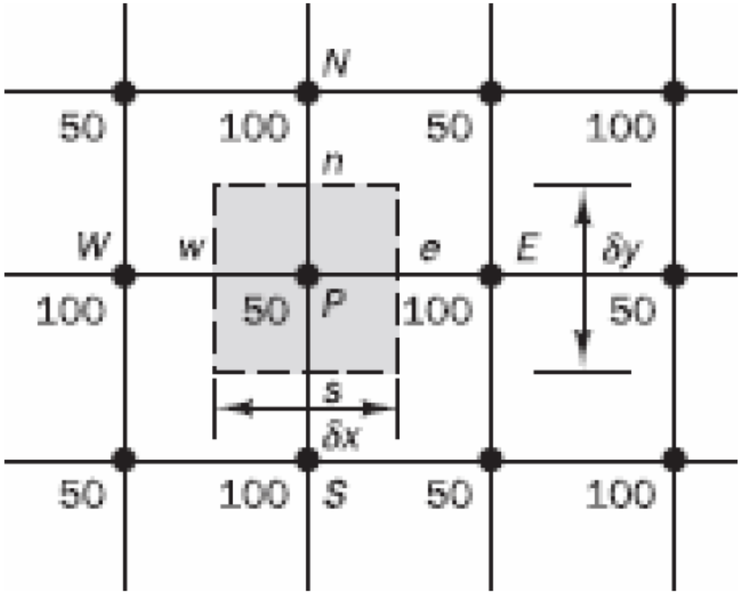
$$\frac{\partial}{\partial x}(\rho u u) + \frac{\partial}{\partial y}(\rho v u) = \frac{\partial}{\partial x}\left(\mu \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) - \frac{\partial p}{\partial x} + S_u$$

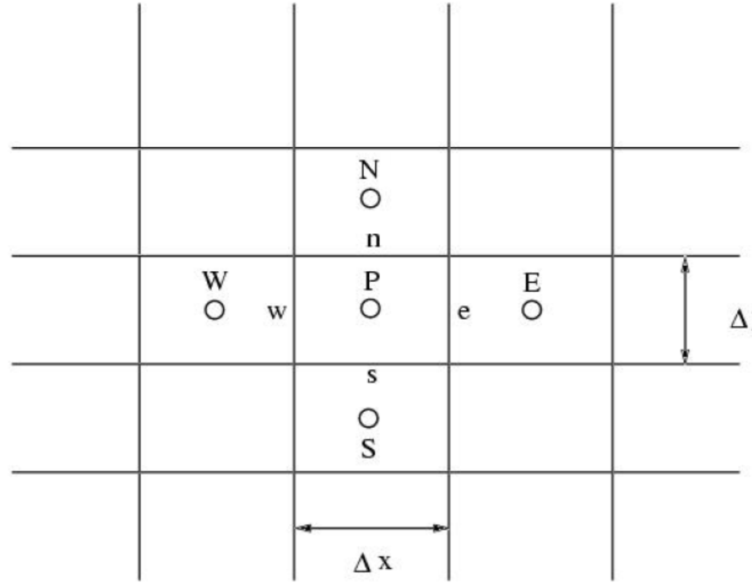
- y-Momentum Equation:

$$\frac{\partial}{\partial x}(\rho u v) + \frac{\partial}{\partial y}(\rho v v) = \frac{\partial}{\partial x}\left(\mu \frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mu \frac{\partial v}{\partial y}\right) - \frac{\partial p}{\partial y} + S_v$$

- Continuity Equation:

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0$$

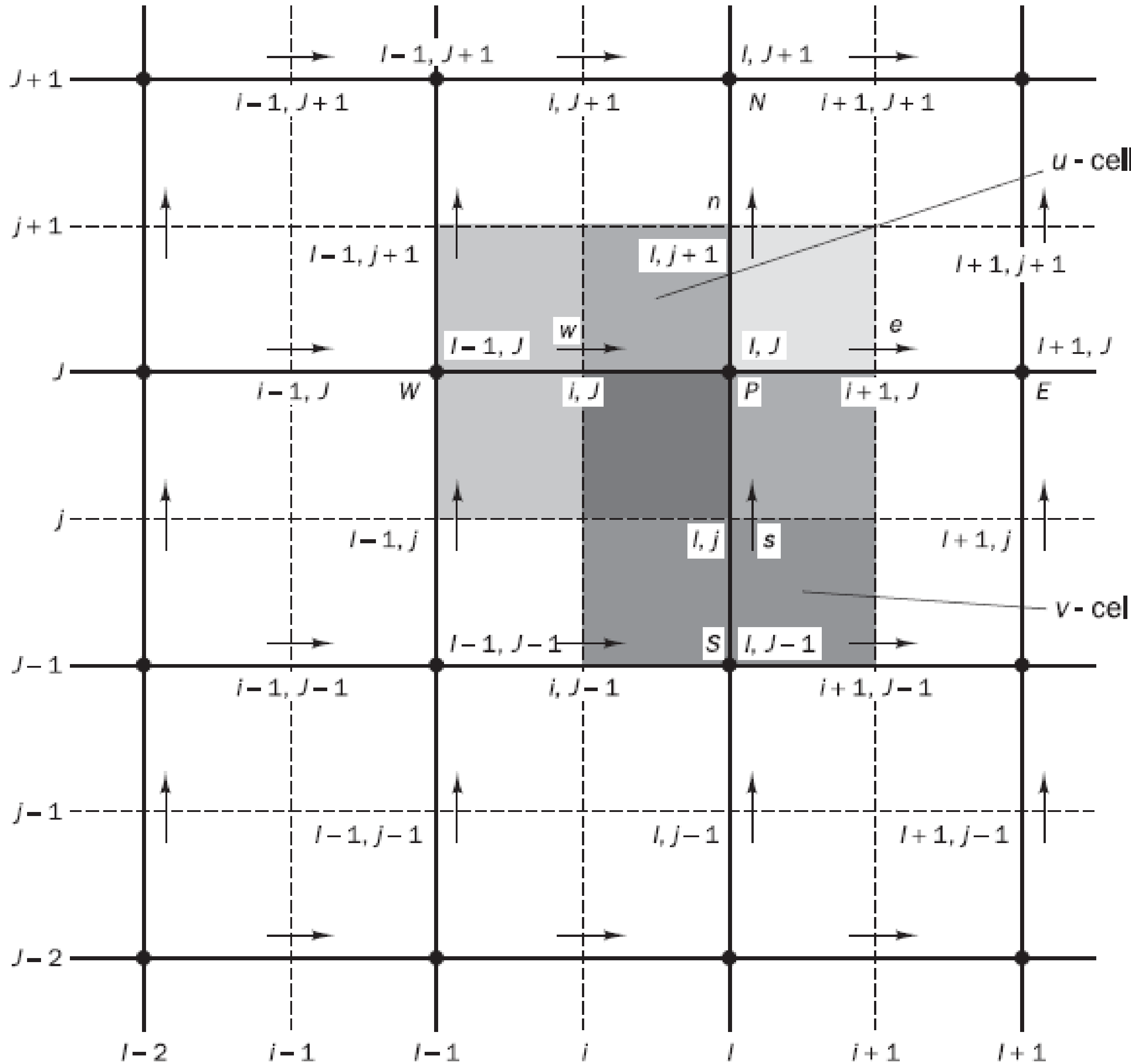




- ∂p/∂x in the *u*- and *v*-momentum equations:

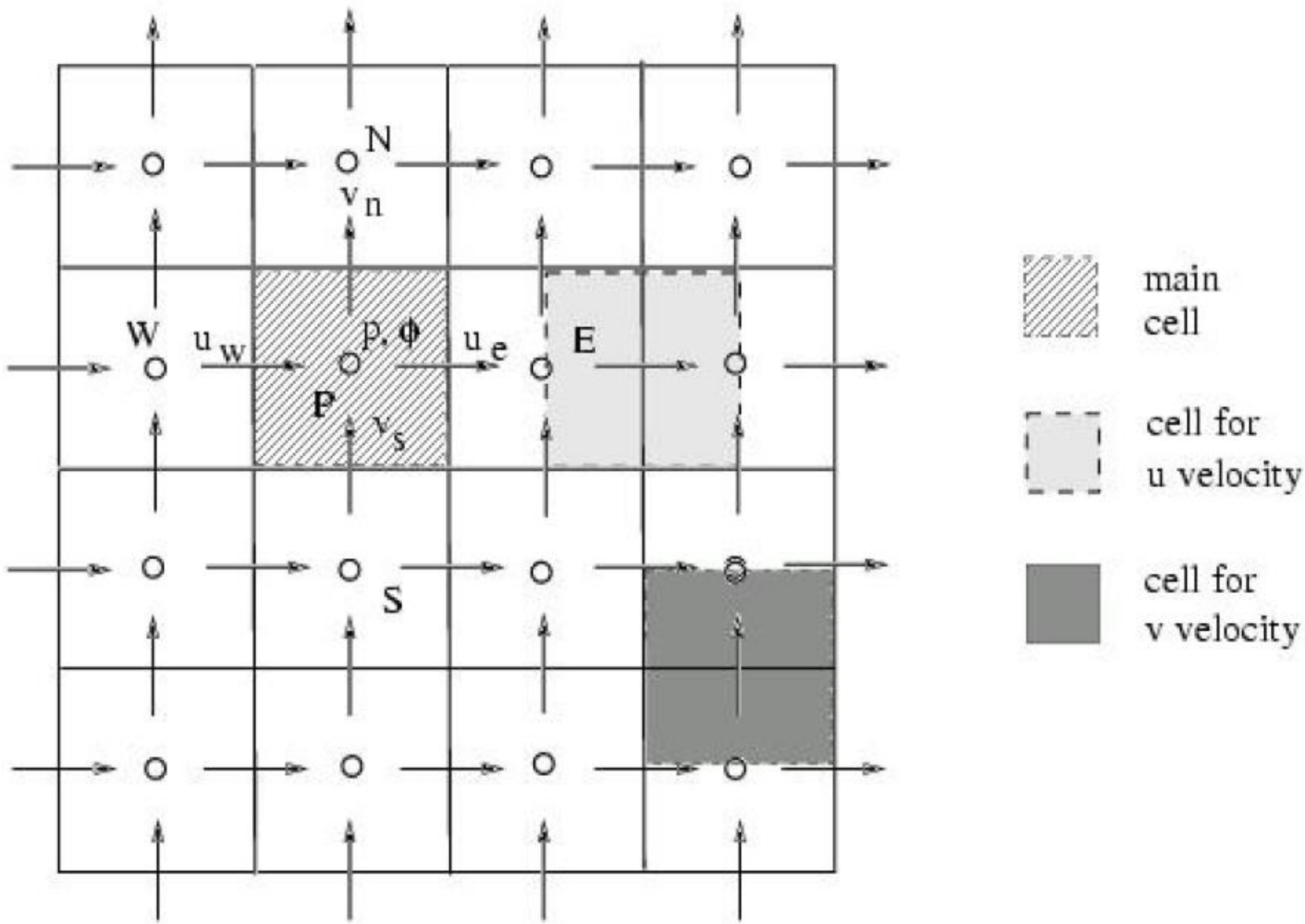
$$\frac{\partial p}{\partial x} = \frac{p_e - p_w}{\delta x} = \frac{\left(\frac{p_E + p_P}{2}\right) - \left(\frac{p_P + p_W}{2}\right)}{\delta x} = \frac{p_E - p_W}{2\delta x} \qquad \frac{\partial p}{\partial y} = \frac{p_N - p_S}{2\delta y}$$

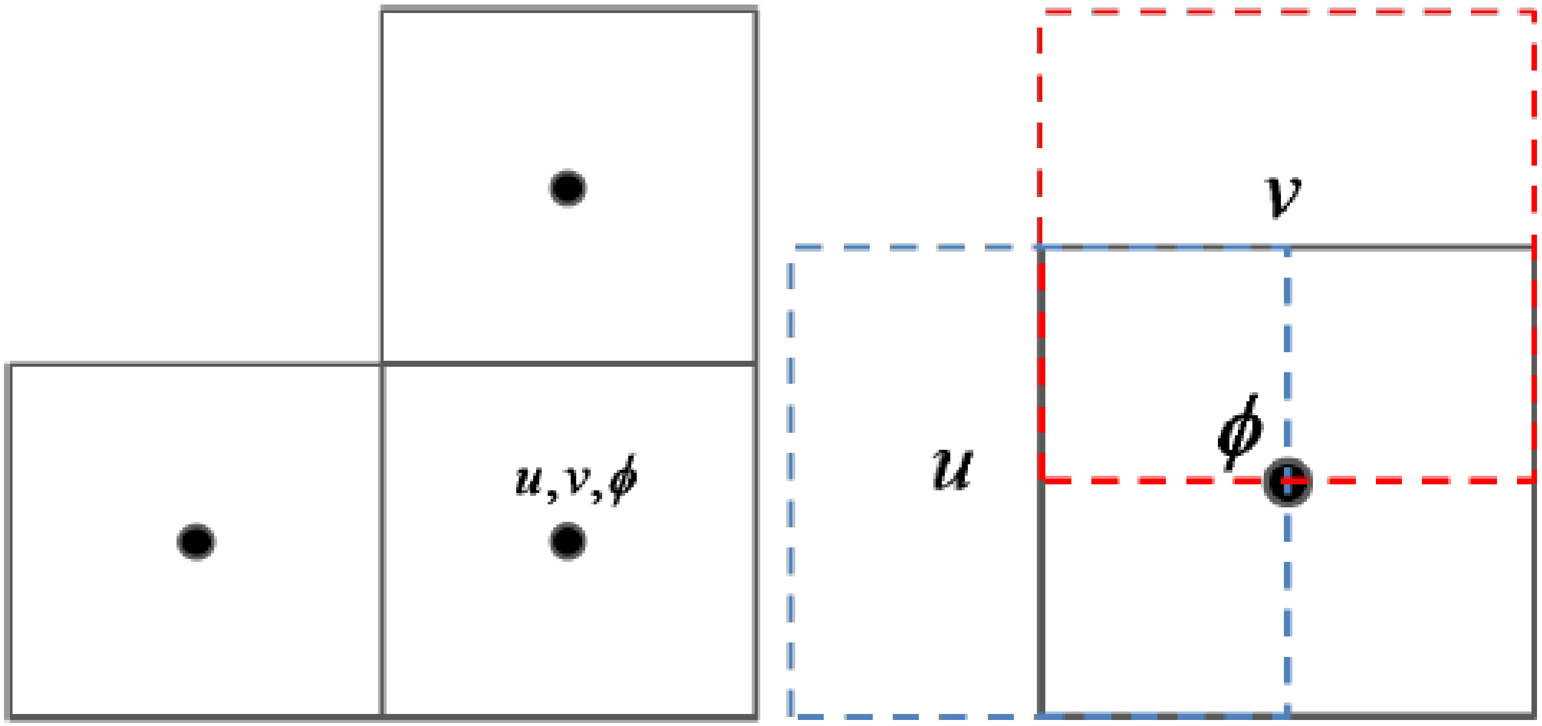
- The pressure at the central node (P) ?
- Give zero momentum source in the discretised equations as a uniform pressure field. This behaviour is obviously non-physical



I − 1, J + 1
I, J + 1
J + 1
i − 1, J + 1
i, J + 1
N
i + 1, J + 1
u - cell
n
j + 1
I − 1, j + 1
I, j + 1
I + 1, j + 1
w
e
I − 1, J
I, J
I + 1, J
J
i − 1, J
W
i, J
P
i + 1, J
E
j
I − 1, j
I, j
s
I + 1, j
v - cell
I − 1, J − 1
S
I, J − 1
J − 1
i − 1, J − 1
i, J − 1
i + 1, J − 1
j − 1
I − 1, j − 1
I, j − 1
I + 1, j − 1
J − 2
I − 2
i − 1
I − 1
i
I
i + 1
I + 1

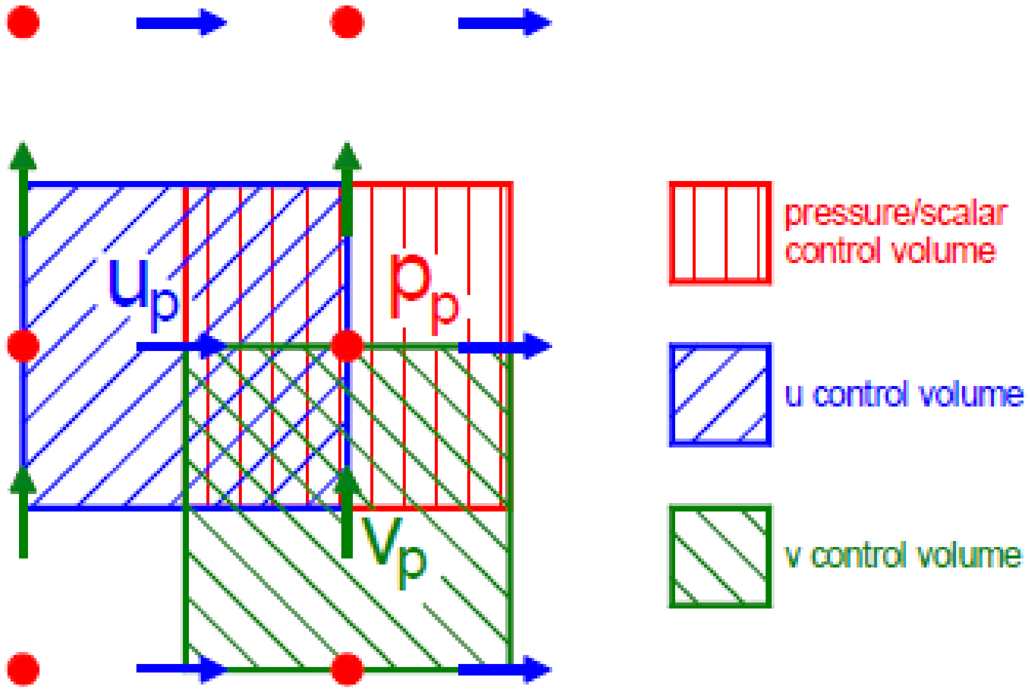
- Store pressure at main cell centroids
- Store velocities on staggered control volumes



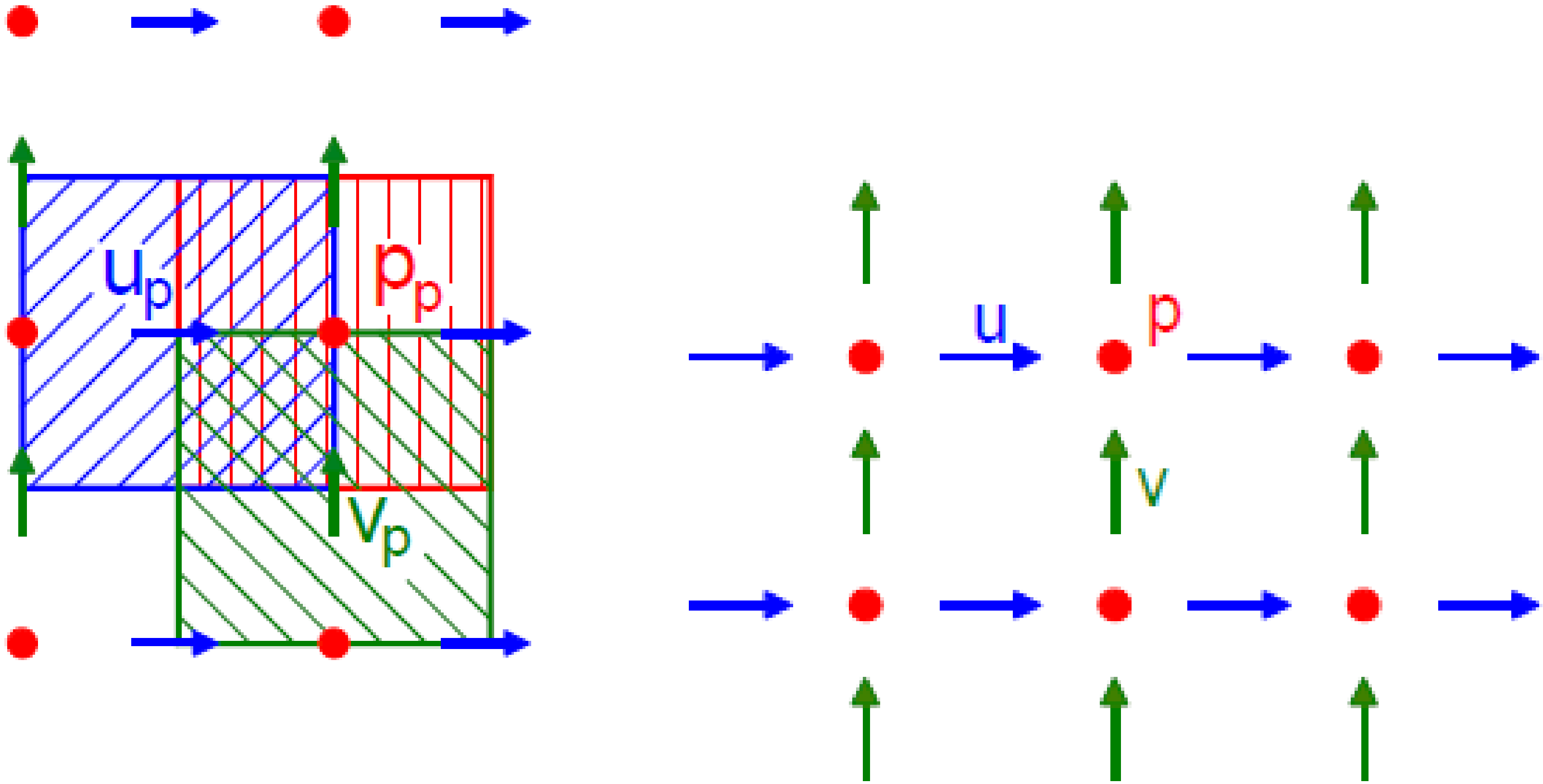


(a) Collocated grid.

(b) Staggered grid.



u_p
p_p
v_p
pressure/scalar control volume
u control volume
v control volume

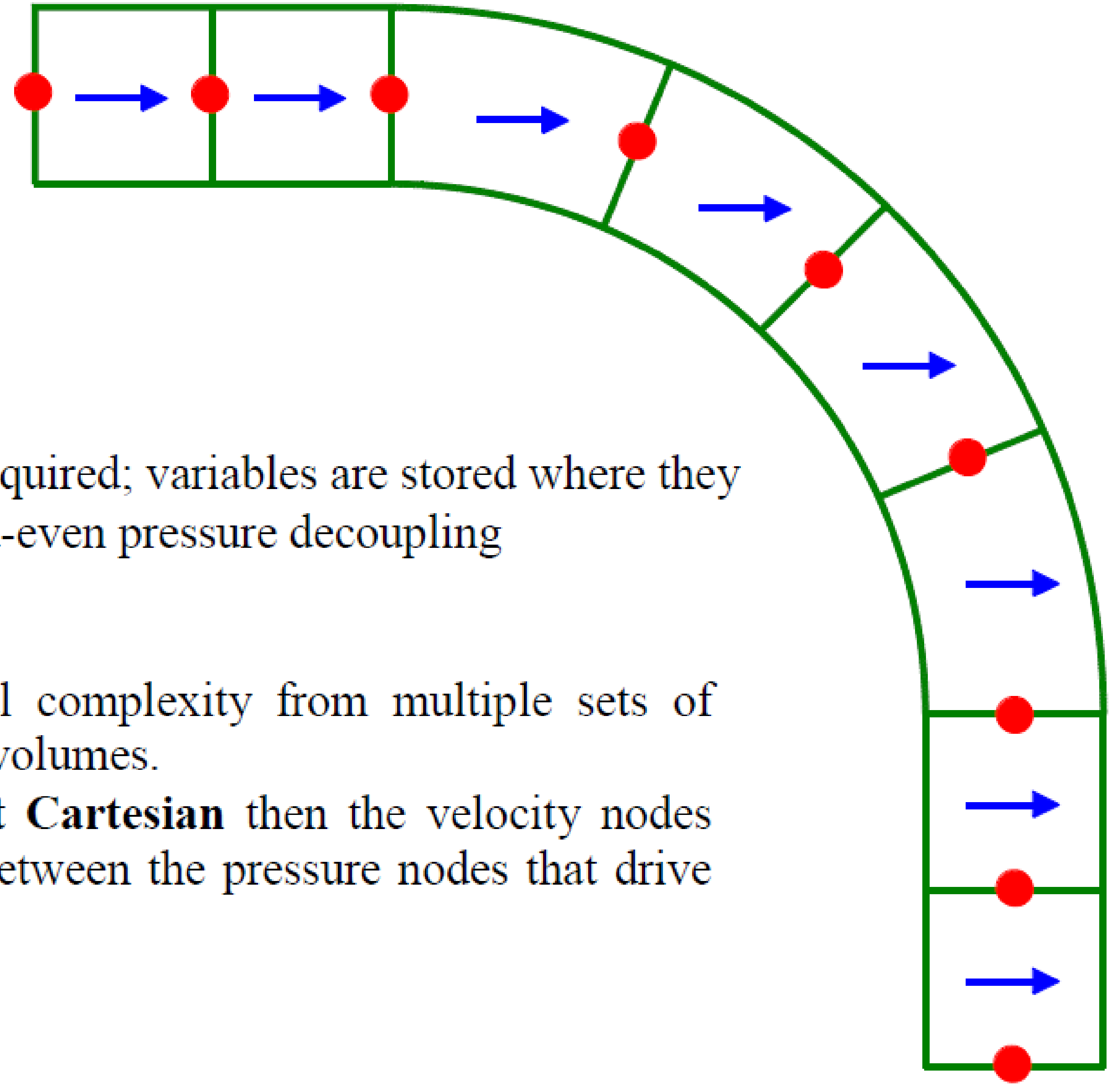

u_p
p_p
v_p
u
p
v

Advantages

- No interpolation required; variables are stored where they
- No problem of odd-even pressure decoupling

Disadvantages

- Added geometrical complexity from multiple sets of nodes and control volumes.
- If the mesh is **not Cartesian** then the velocity nodes may cease to lie between the pressure nodes that drive them (see right).

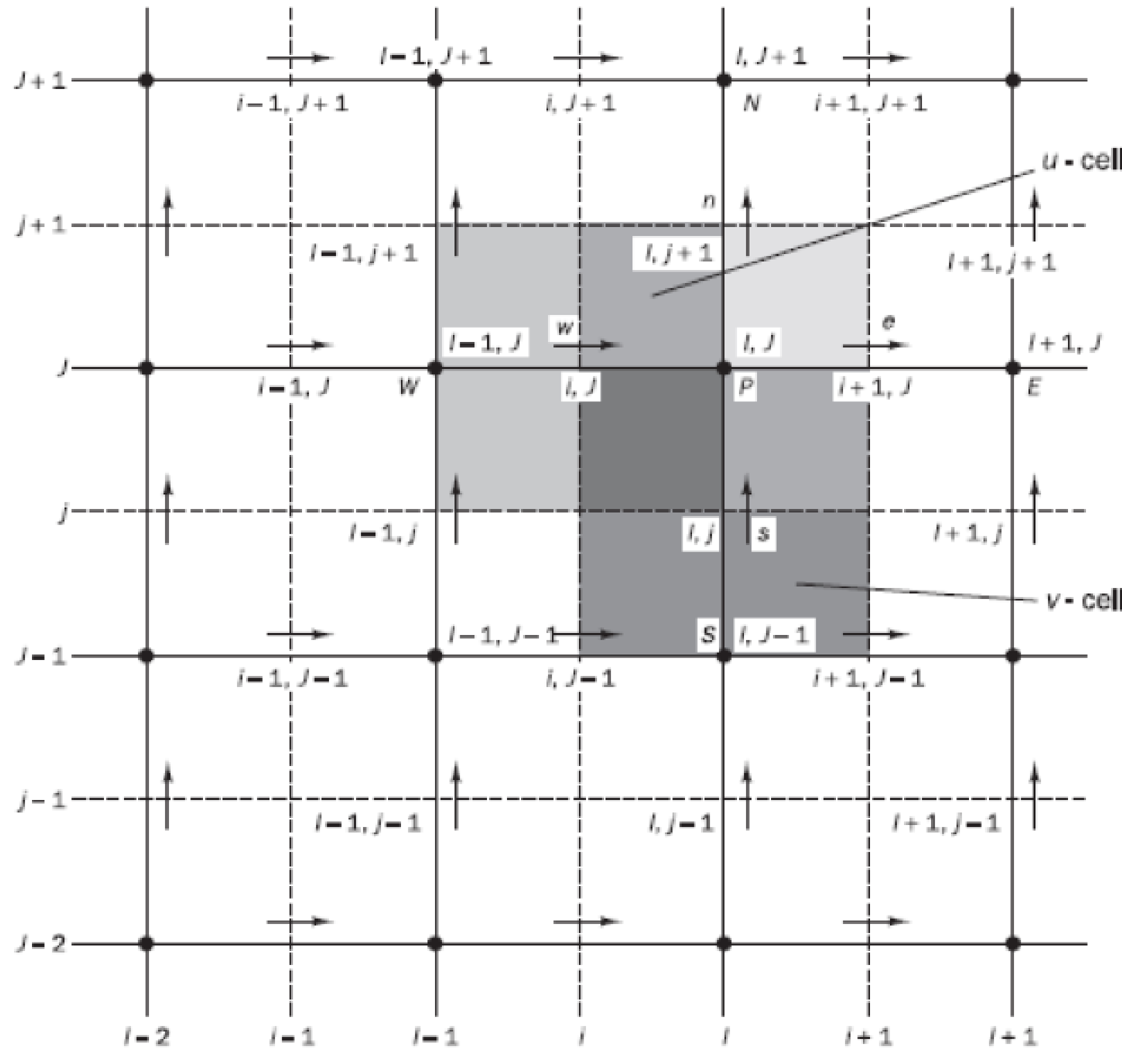


Staggered Grid

- Since the velocity grid is staggered the new notation based on grid line and cell face numbering will be used.
- Forward or Backward Staggered Velocity Grids.
- In Staggered Grid:

$$\frac{\partial p}{\partial x} = \frac{p_P - p_W}{\delta x_u}$$

$$\frac{\partial p}{\partial y} = \frac{p_P - p_S}{\delta y_v}$$



Momentum Equation

- The uniform grids in Figure are backward staggered since the i-location for the u-velocity $u_{i,J}$ is at a distance of $-\delta x_u/2$ from the scalar node (I , J).
- In the new co-ordinate system the discretised u-momentum equation for the velocity at location (i , J) is given by:

$$a_{i,J}u_{i,J} = \sum a_{nb}u_{nb} - \frac{p_{I,J} - p_{I-1,J}}{\delta x_u}\Delta V_u + \bar{S}\Delta V_u$$

$$a_{i,J}u_{i,J} = \sum a_{nb}u_{nb} + (p_{I-1,J} - p_{I,J})A_{i,J} + b_{i,J} \qquad a_e u_e = \sum_{nb} a_{nb}u_{nb} + \Delta y(p_P - p_E) + b_e$$

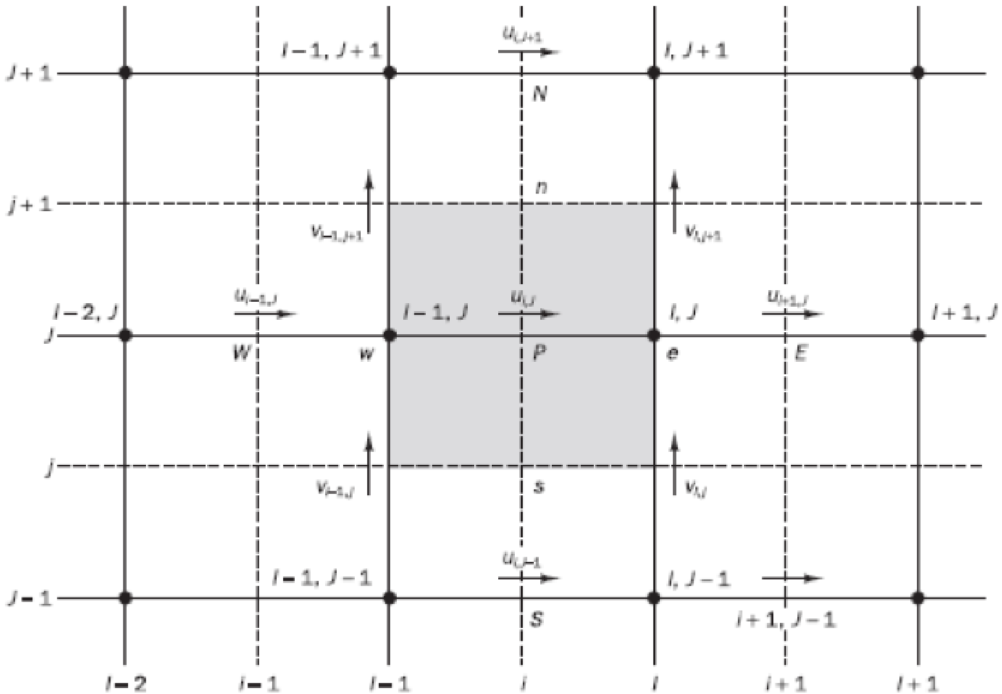
- where ΔV_u is the volume of the u-cell, $b_{i,J} = \hat{S}\Delta V_u$ is the momentum source term, $A_{i,J}$ is the (east or west) cell face area of the u-control volume.

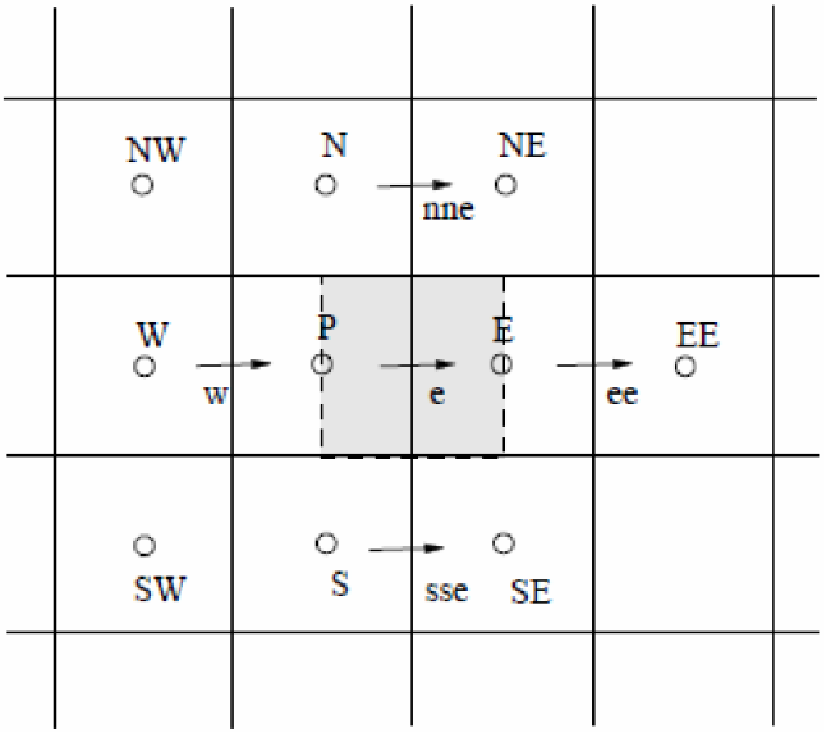
$$a_{i,J}u_{i,J} = \sum a_{nb}u_{nb} - \frac{p_{I,J} - p_{I-1,J}}{\delta x_u}\Delta V_u + \bar{S}\Delta V_u \tag{6.8}$$

$$a_{i,J}u_{i,J} = \sum a_{nb}u_{nb} + (p_{I-1,J} - p_{I,J})A_{i,J} + b_{i,J}$$

In the new numbering system the E, W, N and S neighbours involved in the summation $\sum a_{nb} u_{nb}$ are $(i-1, J)$, $(i+1, J)$, $(i, J-1)$ and $(i, J+1)$. Their locations and the prevailing velocities are shown in more detail in Figure 6.3. The values of coefficients $a_{i,J}$ and a_{nb} may be calculated with any of the differencing methods (upwind, hybrid, QUICK, TVD) suitable for convection–diffusion problems. The coefficients contain combinations of the convective flux per unit mass F and the diffusive conductance D at u-control volume cell faces. Applying the new notation system we give the values of F and D for each of the faces e, w, n and s of the u-control volume:

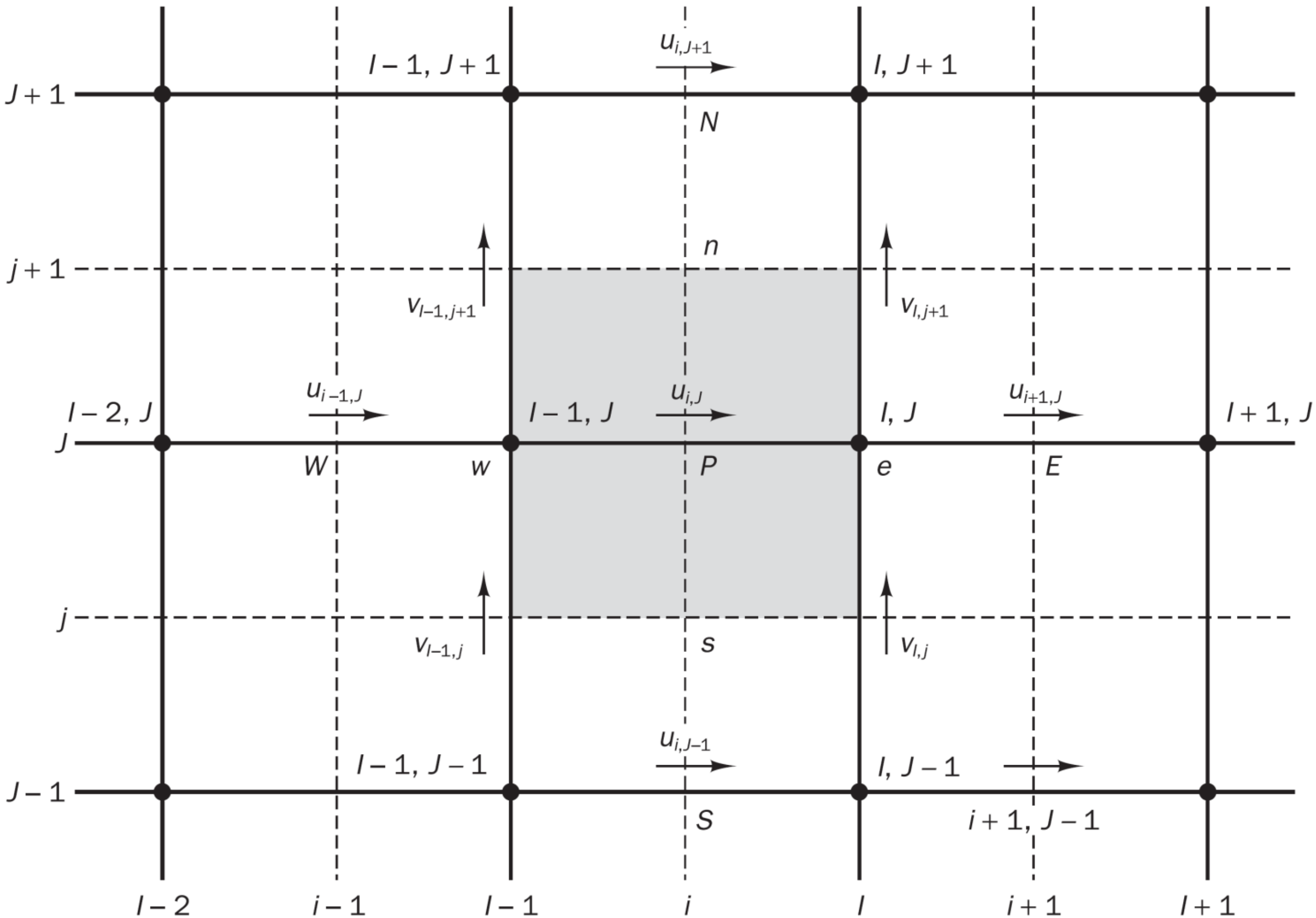
u-Control Volume





$$a_{i,J}u_{i,J} = \sum a_{nb}u_{nb} + (p_{I-1,J} - p_{I,J})A_{i,J} + b_{i,J}$$

$$a_e u_e = \sum_{\mathrm{nb}} a_{\mathrm{nb}} u_{\mathrm{nb}} + \Delta y (p_P - p_E) + b_e$$



$$F_w = (\rho u)_w = \frac{F_{i,J} + F_{i-1,J}}{2}$$

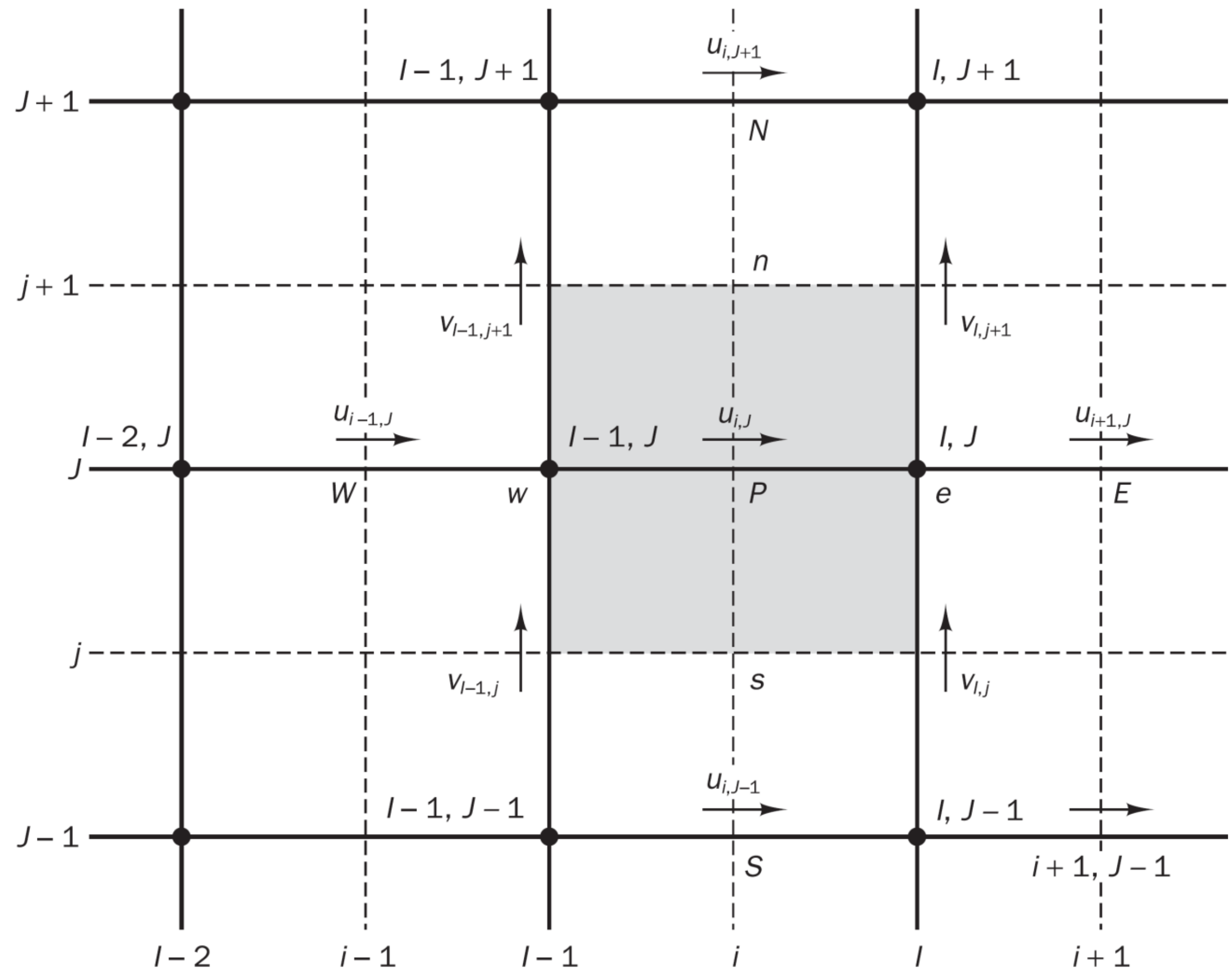
$$= \frac{1}{2}\left[\left(\frac{\rho_{I,J} + \rho_{I-1,J}}{2}\right)u_{i,J} + \left(\frac{\rho_{I-1,J} + \rho_{I-2,J}}{2}\right)u_{i-1,J}\right] \qquad (6.9a)$$

$$F_w = (\rho u)_w = \frac{F_{i,J} + F_{i-1,J}}{2} = \frac{1}{2}\left[\left(\frac{\rho_{I,J} + \rho_{I-1,J}}{2}\right)u_{i,J} + \left(\frac{\rho_{I-1,J} + \rho_{I-2,J}}{2}\right)u_{i-1,J}\right]$$

$$F_e = (\rho u)_e = \frac{F_{i+1,J} + F_{i,J}}{2} = \frac{1}{2}\left[\left(\frac{\rho_{I+1,J} + \rho_{I,J}}{2}\right)u_{i+1,J} + \left(\frac{\rho_{I,J} + \rho_{I-1,J}}{2}\right)u_{i,J}\right]$$

$$F_s = (\rho v)_s = \frac{F_{I,j} + F_{I-1,j}}{2} = \frac{1}{2}\left[\left(\frac{\rho_{I,J} + \rho_{I,J-1}}{2}\right)v_{I,j} + \left(\frac{\rho_{I-1,J} + \rho_{I-1,J-1}}{2}\right)v_{I-1,j}\right]$$

$$F_n = (\rho v)_n = \frac{F_{I,j+1} + F_{I-1,j+1}}{2} = \frac{1}{2}\left[\left(\frac{\rho_{I,J+1} + \rho_{I,J}}{2}\right)v_{I,j+1} + \left(\frac{\rho_{I-1,J+1} + \rho_{I-1,J}}{2}\right)v_{I-1,j+1}\right]$$



$$D_w = \frac{\Gamma_{I-1,J}}{x_i - x_{i-1}}$$

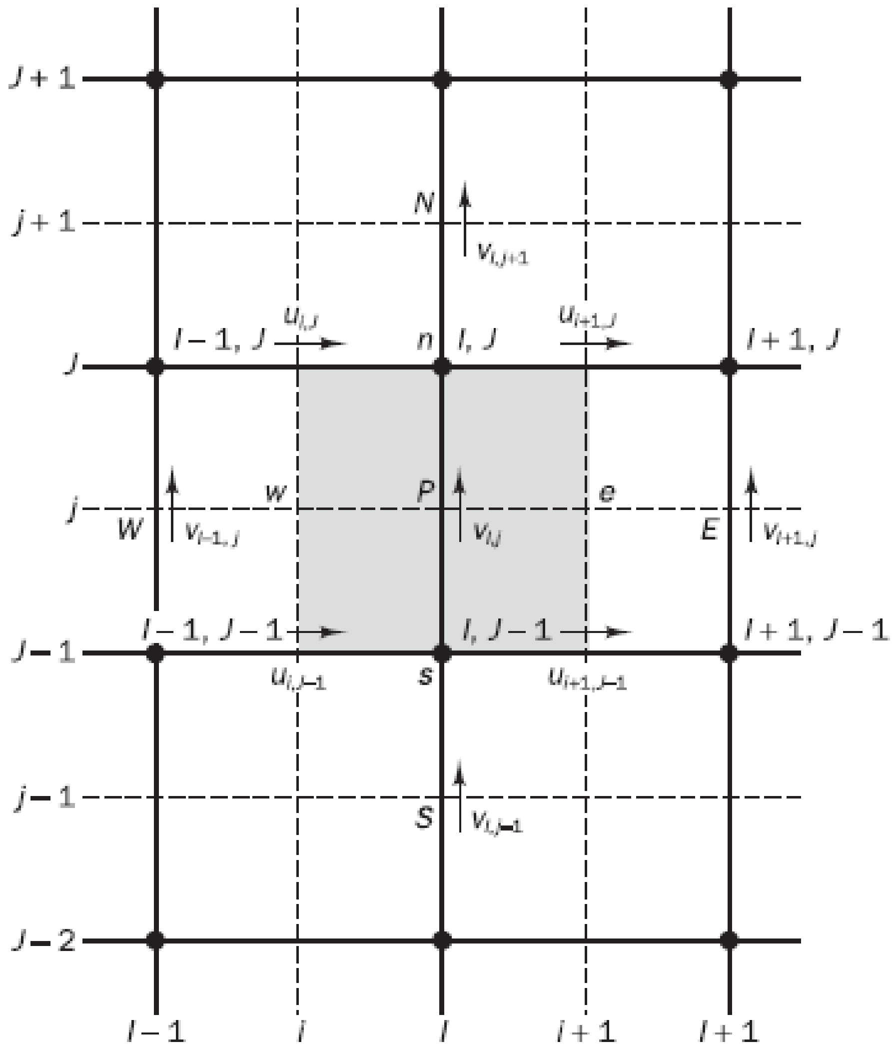
$$D_e = \frac{\Gamma_{I,J}}{x_{i+1} - x_i}$$

$$D_s = \frac{\Gamma_{I-1,J} + \Gamma_{I,J} + \Gamma_{I-1,J-1} + \Gamma_{I,J-1}}{4(y_J - y_{J-1})}$$

$$D_n = \frac{\Gamma_{I-1,J+1} + \Gamma_{I,J+1} + \Gamma_{I-1,J} + \Gamma_{I,J}}{4(y_{J+1} - y_J)}$$

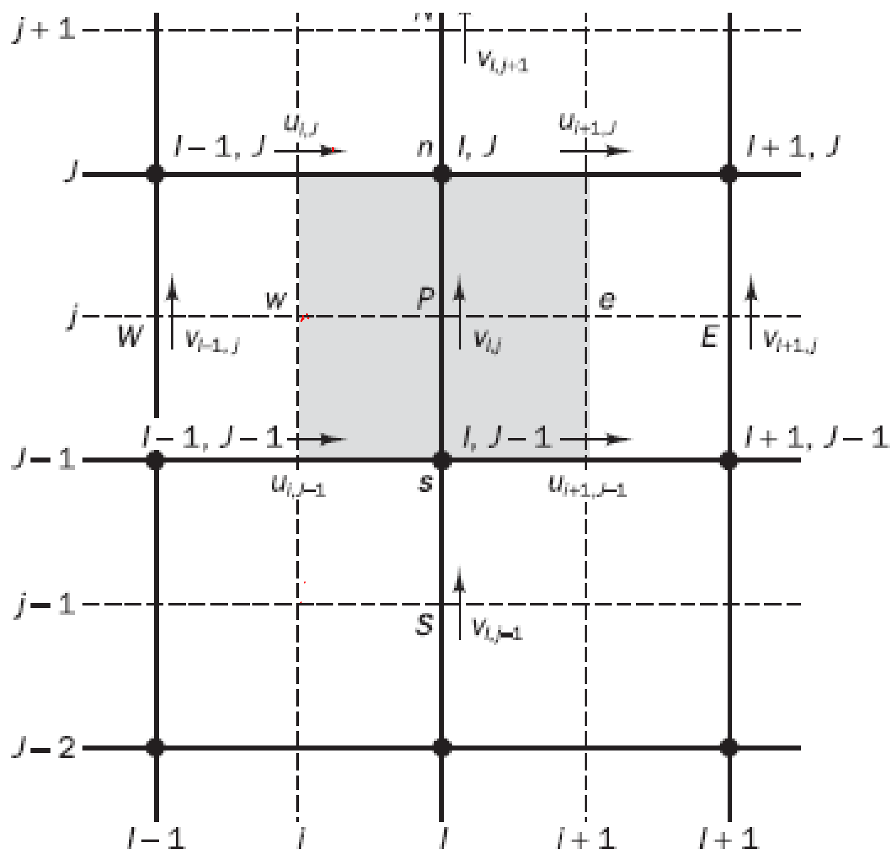
$$a_{I,j} v_{I,j} = \sum a_{nb} v_{nb} + (p_{I,J-1} - p_{I,J}) A_{I,j} + b_{I,j} \tag{6.10}$$

The neighbours involved in the summation $\sum a_{nb} v_{nb}$ and prevailing velocities are as shown in Figure 6.4.



$$F_w = (\rho u)_w = \frac{F_{i,J} + F_{i,J-1}}{2}$$

$$= \frac{1}{2}\left[\left(\frac{\rho_{I,J} + \rho_{I-1,J}}{2}\right)u_{i,J} + \left(\frac{\rho_{I-1,J-1} + \rho_{I,J-1}}{2}\right)u_{i,J-1}\right] \qquad (6.11a)$$



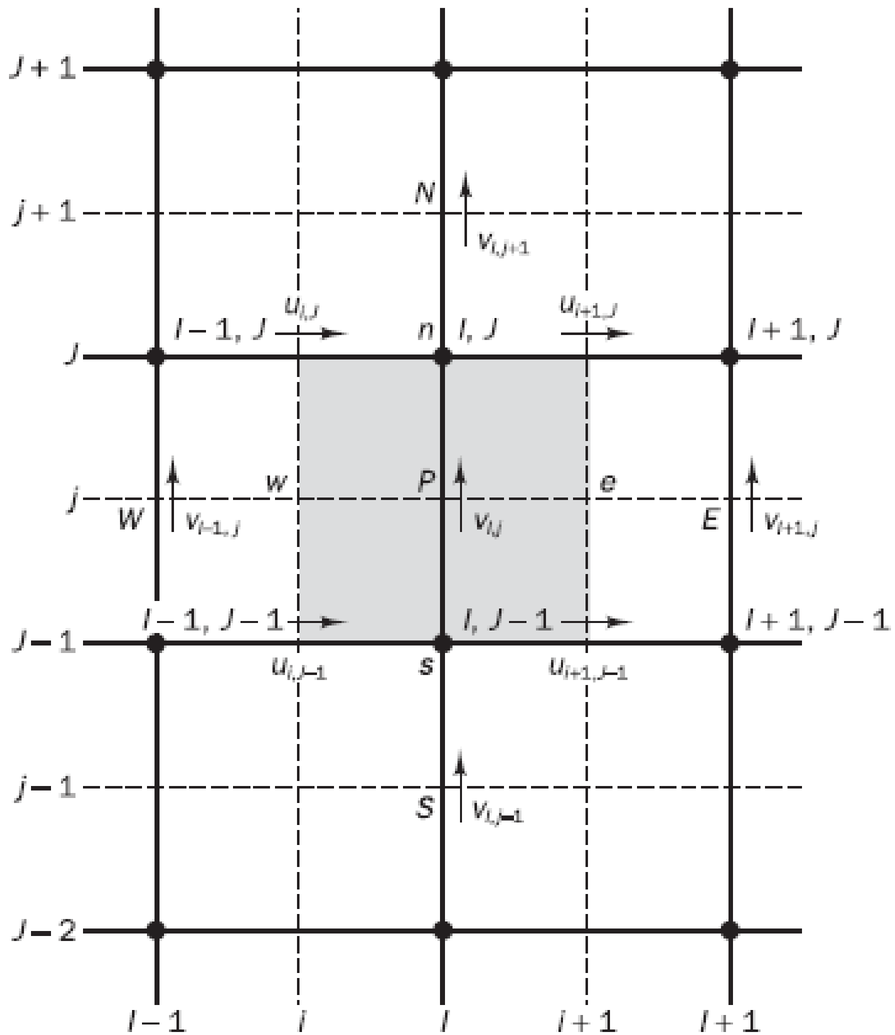
$$F_w = (\rho u)_w = \frac{F_{i,J} + F_{i,J-1}}{2} = \frac{1}{2}\left[\left(\frac{\rho_{I,J} + \rho_{I-1,J}}{2} \right) u_{i,J} + \left(\frac{\rho_{I-1,J-1} + \rho_{I,J-1}}{2} \right) u_{i,J-1} \right]$$

$$F_e = (\rho u)_e = \frac{F_{i+1,J} + F_{i+1,J-1}}{2} = \frac{1}{2}\left[\left(\frac{\rho_{I+1,J} + \rho_{I,J}}{2} \right) u_{i+1,J} + \left(\frac{\rho_{I,J-1} + \rho_{I+1,J-1}}{2} \right) u_{i+1,J-1} \right]$$

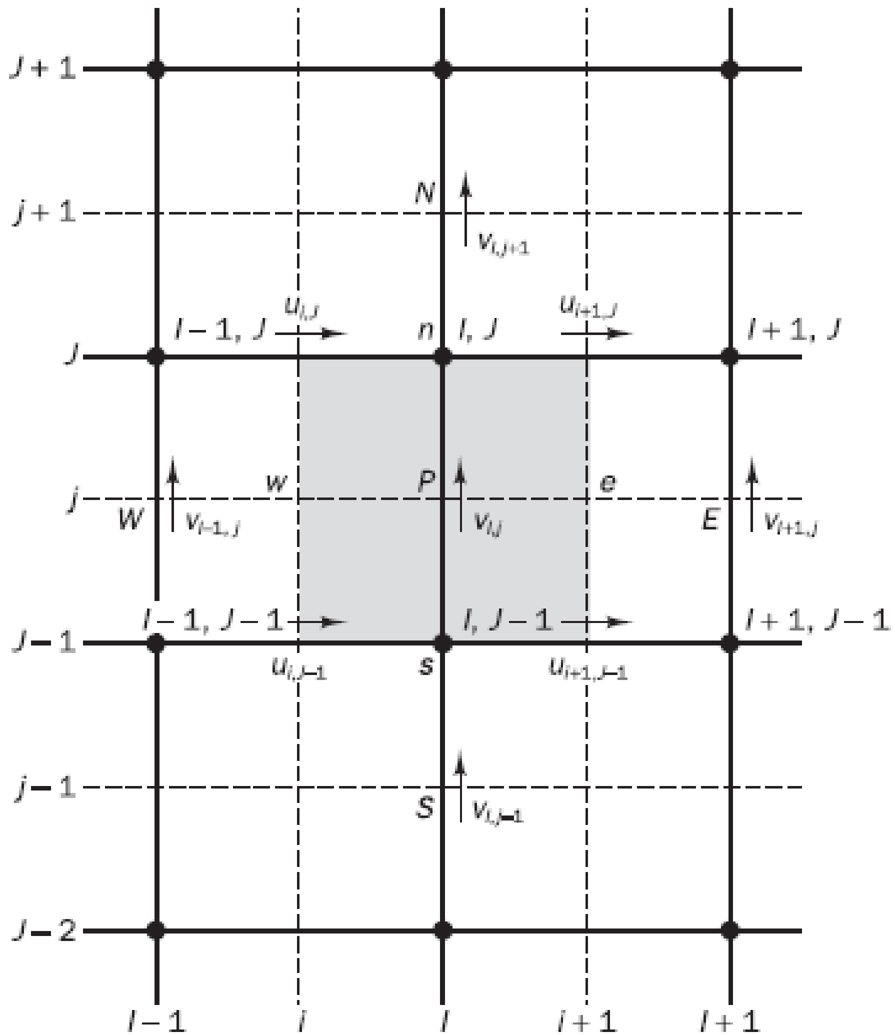
$$F_s = (\rho v)_s = \frac{F_{I,j-1} + F_{I,j}}{2} = \frac{1}{2}\left[\left(\frac{\rho_{I,J-1} + \rho_{I,J-2}}{2} \right) v_{I,j-1} + \left(\frac{\rho_{I,J} + \rho_{I,J-1}}{2} \right) v_{I,j} \right]$$

$$F_n = (\rho v)_n = \frac{F_{I,j} + F_{I,j+1}}{2} = \frac{1}{2}\left[\left(\frac{\rho_{I,J} + \rho_{I,J-1}}{2} \right) v_{I,j} + \left(\frac{\rho_{I,J+1} + \rho_{I,J}}{2} \right) v_{I,j+1} \right]$$

$$D_w = \frac{\Gamma_{I-1,J-1} + \Gamma_{I,J-1} + \Gamma_{I-1,J} + \Gamma_{I,J}}{4(x_I - x_{I-1})}$$



$$D_w = \frac{\Gamma_{I-1,J-1} + \Gamma_{I,J-1} + \Gamma_{I-1,J} + \Gamma_{I,J}}{4(x_I - x_{I-1})}$$



$$D_s = \frac{\Gamma_{I,J-1}}{y_j - y_{j-1}}$$

$$D_n = \frac{\Gamma_{I,J}}{y_{j+1} - y_j}$$

$$D_w = \frac{\Gamma_{I-1,J-1} + \Gamma_{I,J-1} + \Gamma_{I-1,J} + \Gamma_{I,J}}{4(x_I - x_{I-1})}$$

$$D_e = \frac{\Gamma_{I,J-1} + \Gamma_{I+1,J-1} + \Gamma_{I,J} + \Gamma_{I+1,J}}{4(x_{I+1} - x_I)}$$

$$D_s = \frac{\Gamma_{I,J-1}}{y_j - y_{j-1}}$$

$$D_n = \frac{\Gamma_{I,J}}{y_{j+1} - y_j}$$

- All properties, such as density and Γ, are stored at the main grid points.
- Continuity: $(\rho u)_e \Delta y - (\rho u)_w \Delta y + (\rho v)_n \Delta x - (\rho v)_s \Delta x = 0$
- u-Momentum: $a_e u_e = \sum_{\mathrm{nb}} a_{\mathrm{nb}} u_{\mathrm{nb}} + \Delta y (p_P - p_E) + b_e$
- v-Momentum: $a_n v_n = \sum_{\mathrm{nb}} a_{\mathrm{nb}} v_{\mathrm{nb}} + \Delta x (p_P - p_N) + b_n$

The SIMPLE algorithm

The acronym SIMPLE stands for Semi-Implicit Method for Pressure-Linked Equations. The algorithm was originally put forward by Patankar and Spalding (1972) and is essentially a guess-and-correct procedure for the

The SIMPLE algorithm

$$a_{i,J}u_{i,J} = \sum a_{nb}u_{nb} + (p_{I-1,J} - p_{I,J})A_{i,J} + b_{i,J} \tag{6.8}$$

$$a_{I,j}v_{I,j} = \sum a_{nb}v_{nb} + (p_{I,J-1} - p_{I,J})A_{I,j} + b_{I,j} \tag{6.10}$$

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To initiate the SIMPLE calculation process a pressure field p^* is guessed. **Discretised momentum equations** (6.8) and (6.10) **are solved** using the guessed pressure field to yield velocity components u^* and v^* as follows:

$$\boxed{a_{i,J}u^*_{i,J} = \sum a_{nb}u^*_{nb} + (p^*_{I-1,J} - p^*_{I,J})A_{i,J} + b_{i,J}} \tag{6.12}$$

$$\boxed{a_{I,j}v^*_{I,j} = \sum a_{nb}v^*_{nb} + (p^*_{I,J-1} - p^*_{I,J})A_{I,j} + b_{I,j}} \tag{6.13}$$

Now we define the correction p' as the difference between correct pressure field p and the guessed pressure field p^*, so that

$$\boxed{p = p^* + p'} \tag{6.14}$$

Similarly we define velocity corrections u' and v' to relate the correct velocities u and v to the guessed velocities u^* and v^*:

$$\boxed{u = u^* + u'} \tag{6.15}$$

$$\boxed{v = v^* + v'} \tag{6.16}$$

Subtraction of equations (6.12) and (6.13) from (6.8) and (6.10), respectively, gives

$$a_{i,J}u_{i,J} = \sum a_{nb}u_{nb} + (p_{I-1,J} - p_{I,J})A_{i,J} + b_{i,J} \tag{6.8}$$

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$$a_{i,J}(u_{i,J} - u^*_{i,J}) = \sum a_{nb}(u_{nb} - u^*_{nb}) + [(p_{I-1,J} - p^*_{I-1,J}) - (p_{I,J} - p^*_{I,J})]A_{i,J} \tag{6.17}$$

$$a_{I,j}(v_{I,j} - v^*_{I,j}) = \sum a_{nb}(v_{nb} - v^*_{nb}) + [(p_{I,J-1} - p^*_{I,J-1}) - (p_{I,J} - p^*_{I,J})]A_{I,j} \tag{6.18}$$

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Using correction formulae (6.14)–(6.16) the equations (6.17)–(6.18) may be rewritten as follows:

$$a_{i,J}u'_{i,J} = \sum a_{nb}u'_{nb} + (p'_{I-1,J} - p'_{I,J})A_{i,J} \quad (6.19)$$

$$a_{I,j}v'_{I,j} = \sum a_{nb}v'_{nb} + (p'_{I,J-1} - p'_{I,J})A_{I,j} \quad (6.20)$$

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At this point an approximation is introduced: $\sum a_{nb}u'_{nb}$ and $\sum a_{nb}v'_{nb}$ are dropped to simplify equations (6.19) and (6.20) for the velocity corrections. **Omission of these terms is the main approximation** of the SIMPLE algorithm. We obtain

$$u'_{i,J} = d_{i,J}(p'_{I-1,J} - p'_{I,J}) \quad (6.21)$$

$$v'_{I,j} = d_{I,j}(p'_{I,J-1} - p'_{I,J}) \quad (6.22)$$

where $d_{i,J} = \dfrac{A_{i,J}}{a_{i,J}}$ and $d_{I,j} = \dfrac{A_{I,j}}{a_{I,j}}$ (6.23)

Equations (6.21) and (6.22) describe the corrections to be applied to velocities through formulae (6.15) and (6.16), which gives

$$\boxed{u_{i,J} = u^*_{i,J} + d_{i,J}(p'_{I-1,J} - p'_{I,J})} \tag{6.24}$$

$$\boxed{v_{I,j} = v^*_{I,j} + d_{I,j}(p'_{I,J-1} - p'_{I,J})} \tag{6.25}$$

Similar expressions exist for $u_{i+1,J}$ and $v_{I,j+1}$:

$$u_{i+1,J} = u^*_{i+1,J} + d_{i+1,J}(p'_{I,J} - p'_{I+1,J}) \tag{6.26}$$

$$v_{I,j+1} = v^*_{I,j+1} + d_{I,j+1}(p'_{I,J} - p'_{I,J+1}) \tag{6.27}$$

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$$[(\rho uA)_{i+1,J} - (\rho uA)_{i,J}] + [(\rho vA)_{I,j+1} - (\rho vA)_{I,j}] = 0 \tag{6.29}$$

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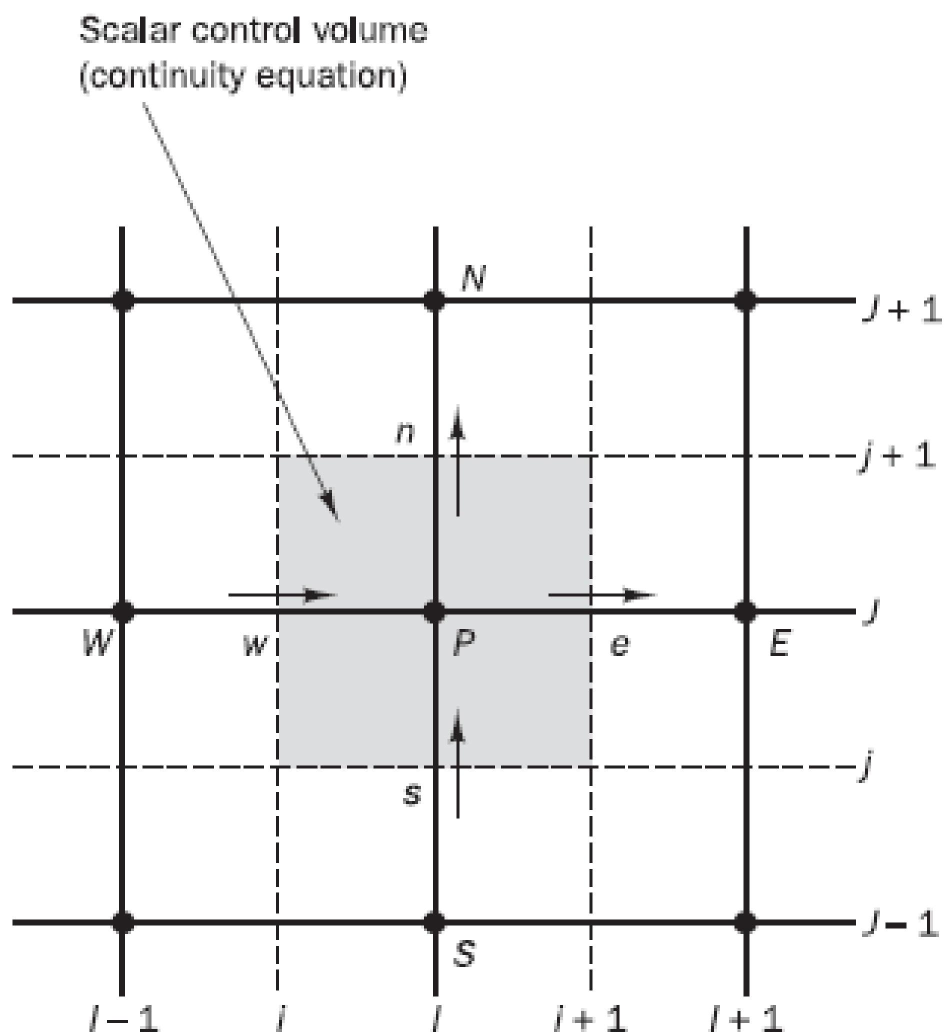

$$[(\rho uA)_{i+1,J} - (\rho uA)_{i,J}] + [(\rho vA)_{I,j+1} - (\rho vA)_{I,j}] = 0 \tag{6.29}$$

Substitution of the corrected velocities of equations (6.24)–(6.27) into discretised continuity equation (6.29) gives

$$\begin{aligned}
&[\rho_{i+1,J}A_{i+1,J}(u^*_{i+1,J} + d_{i+1,J}(p'_{I,J} - p'_{I+1,J})) - \rho_{i,J}A_{i,J}(u^*_{i,J} \\
&\quad + d_{i,J}(p'_{I-1,J} - p'_{I,J}))] + [\rho_{I,j+1}A_{I,j+1}(v^*_{I,j+1} + d_{I,j+1}(p'_{I,J} - p'_{I,J+1})) \\
&\quad - \rho_{I,j}A_{I,j}(v^*_{I,j} + d_{I,j}(p'_{I,J-1} - p'_{I,J}))] = 0
\end{aligned} \tag{6.30}$$

This may be rearranged to give

$$\begin{aligned}
&[(\rho dA)_{i+1,J} + (\rho dA)_{i,J} + (\rho dA)_{I,j+1} + (\rho dA)_{I,j}]p'_{I,J} = (\rho dA)_{i+1,J}p'_{I+1,J} \\
&\quad + (\rho dA)_{i,J}p'_{I-1,J} + (\rho dA)_{I,j+1}p'_{I,J+1} + (\rho dA)_{I,j}p'_{I,J-1} \\
&\quad + [(\rho u^*A)_{i,J} - (\rho u^*A)_{i+1,J} + (\rho v^*A)_{I,j} - (\rho v^*A)_{I,j+1}]
\end{aligned} \tag{6.31}$$

Identifying the coefficients of p', this may be written as

$$\boxed{a_{I,J}p'_{I,J} = a_{I+1,J}p'_{I+1,J} + a_{I-1,J}p'_{I-1,J} + a_{I,J+1}p'_{I,J+1} + a_{I,J-1}p'_{I,J-1} + b'_{I,J}} \tag{6.32}$$

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where $a_{I,J} = a_{I+1,J} + a_{I-1,J} + a_{I,J+1} + a_{I,J-1}$ and the coefficients are given below:

$a_{I+1,J}$	$a_{I-1,J}$	$a_{I,J+1}$	$a_{I,J-1}$	$b'_{I,J}$
$(\rho dA)_{i+1,J}$	$(\rho dA)_{i,J}$	$(\rho dA)_{I,j+1}$	$(\rho dA)_{I,j}$	$(\rho u^* A)_{i,J} - (\rho u^* A)_{i+1,J}$ $+ (\rho v^* A)_{I,j} - (\rho v^* A)_{I,j+1}$

$p^* = p$, $u^* = u$ and $v^* = v$.

The pressure correction equation is susceptible to divergence unless some **under-relaxation** is used during the iterative process, and new, improved, pressures p^{new} are obtained with

$$p^{new} = p^* + \alpha_p p' \tag{6.33}$$

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The velocities are also under-relaxed. The iteratively improved velocity components u^{new} and v^{new} are obtained from

$$u^{new} = \alpha_u u + (1 - \alpha_u)u^{(n-1)} \tag{6.34}$$

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$$\frac{a_{i,J}}{\alpha_u}u_{i,J} = \sum a_{nb}u_{nb} + (p_{I-1,J} - p_{I,J})A_{i,J} + b_{i,J} + \left[(1 - \alpha_u)\frac{a_{i,J}}{\alpha_u}\right]u_{i,J}^{(n-1)} \tag{6.36}$$

and the discretised v-momentum equation

$$\frac{a_{I,j}}{\alpha_v}v_{I,j} = \sum a_{nb}v_{nb} + (p_{I,J-1} - p_{I,J})A_{I,j} + b_{I,j} + \left[(1 - \alpha_v)\frac{a_{I,j}}{\alpha_v}\right]v_{I,j}^{(n-1)} \tag{6.37}$$

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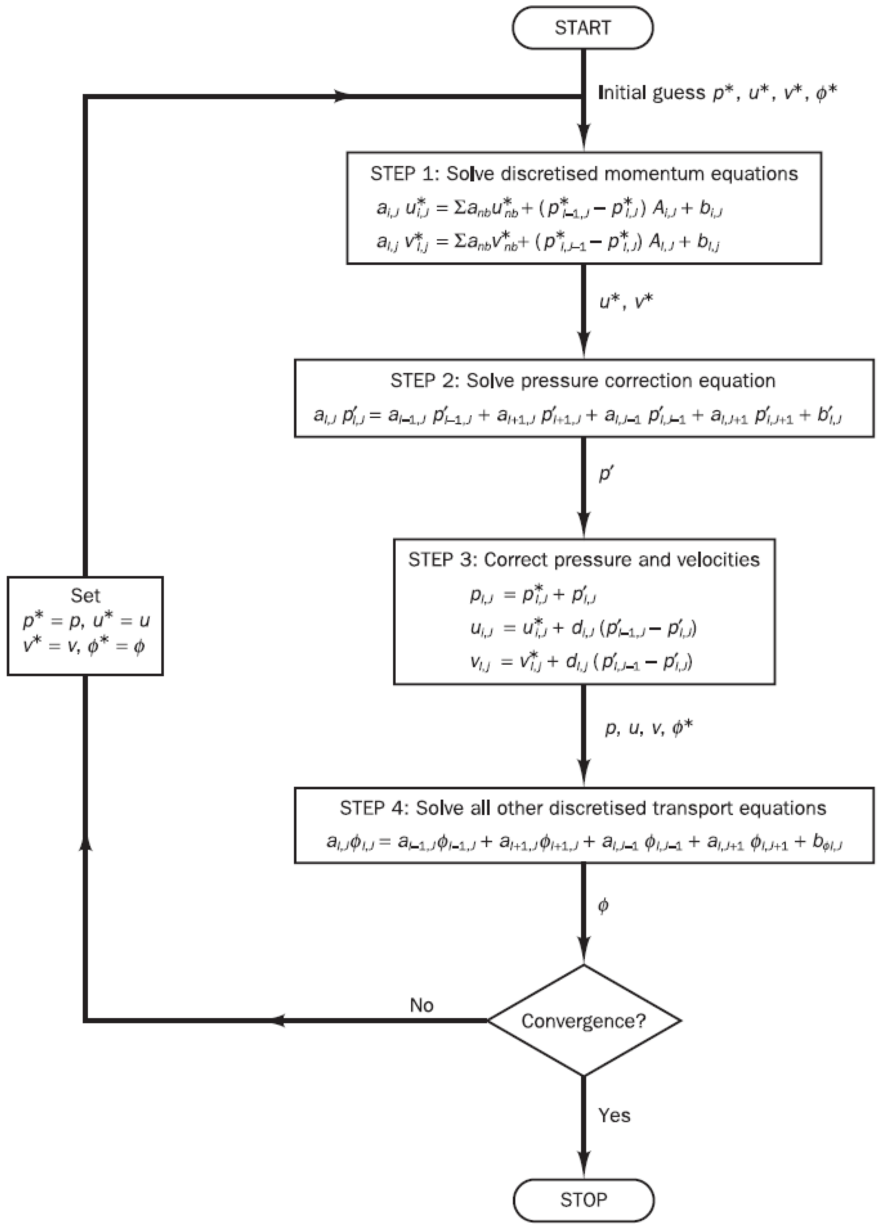
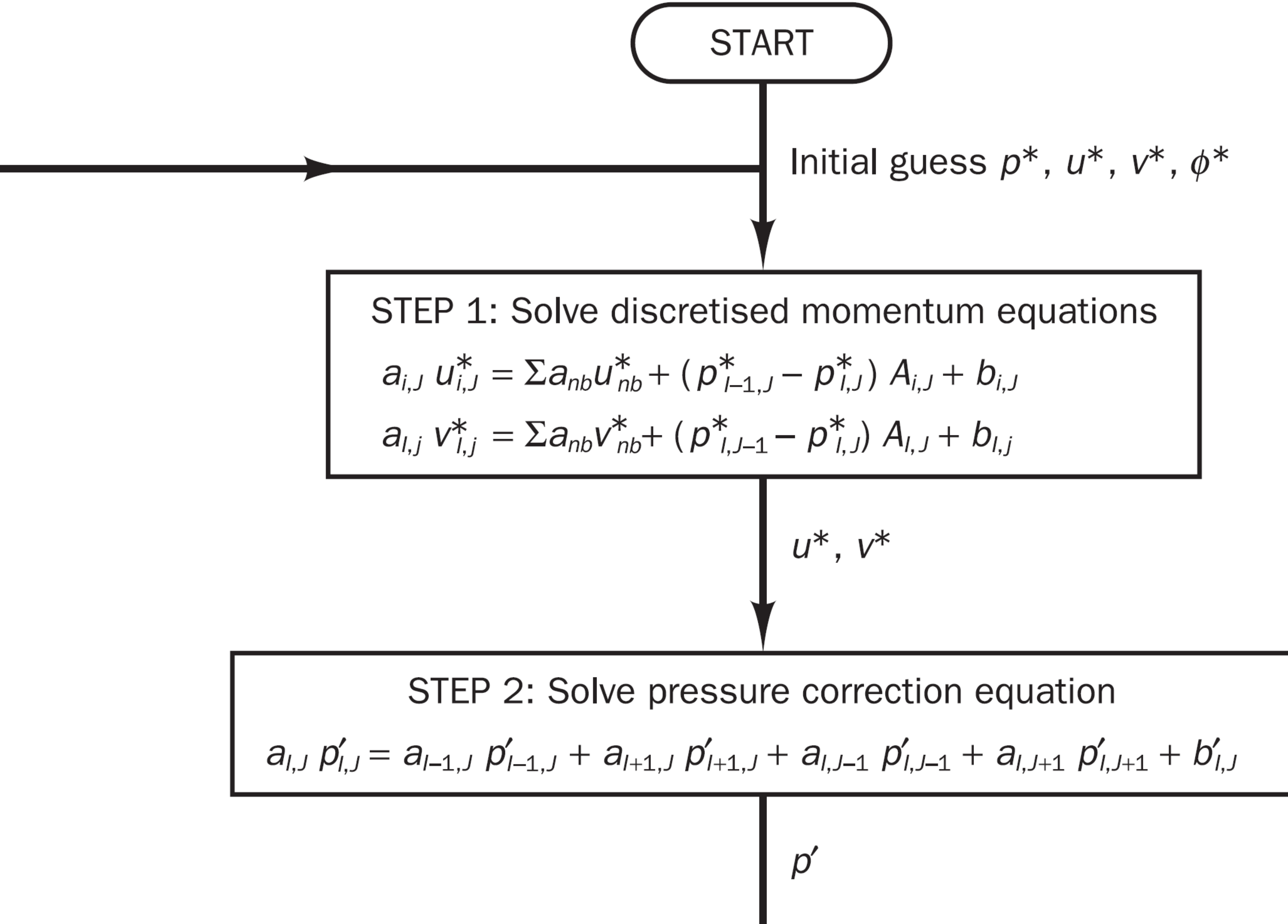
$$d_{i,J} = \frac{A_{i,J}\alpha_u}{a_{i,J}}, \quad d_{i+1,J} = \frac{A_{i+1,J}\alpha_u}{a_{i+1,J}}, \quad d_{I,j} = \frac{A_{I,j}\alpha_v}{a_{I,j}}, \quad d_{I,j+1} = \frac{A_{I,j+1}\alpha_v}{a_{I,j+1}}$$



START
Initial guess p^*, u^*, v^*, ϕ^*
STEP 1: Solve discretised momentum equations
$a_{i,J}\, u^*_{i,J} = \Sigma a_{nb} u^*_{nb} + (p^*_{I-1,J} - p^*_{I,J})\, A_{i,J} + b_{i,J}$
$a_{I,j}\, v^*_{I,j} = \Sigma a_{nb} v^*_{nb} + (p^*_{I,J-1} - p^*_{I,J})\, A_{I,j} + b_{I,j}$
u^*, v^*
STEP 2: Solve pressure correction equation
$a_{I,J}\, p'_{I,J} = a_{I-1,J}\, p'_{I-1,J} + a_{I+1,J}\, p'_{I+1,J} + a_{I,J-1}\, p'_{I,J-1} + a_{I,J+1}\, p'_{I,J+1} + b'_{I,J}$
p'
STEP 3: Correct pressure and velocities
$p_{I,J} = p^*_{I,J} + p'_{I,J}$
$u_{i,J} = u^*_{i,J} + d_{i,J}\,(p'_{I-1,J} - p'_{I,J})$
$v_{I,j} = v^*_{I,j} + d_{I,j}\,(p'_{I,J-1} - p'_{I,J})$
p, u, v, ϕ^*
STEP 4: Solve all other discretised transport equations
$a_{I,J}\phi_{I,J} = a_{I-1,J}\phi_{I-1,J} + a_{I+1,J}\phi_{I+1,J} + a_{I,J-1}\,\phi_{I,J-1} + a_{I,J+1}\,\phi_{I,J+1} + b_{\phi I,J}$
ϕ
Convergence?
No
Yes
STOP
Set
$p^* = p,\ u^* = u$
$v^* = v,\ \phi^* = \phi$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p'

p'

STEP 3: Correct pressure and velocities

$$p_{I,J} = p^*_{I,J} + p'_{I,J}$$

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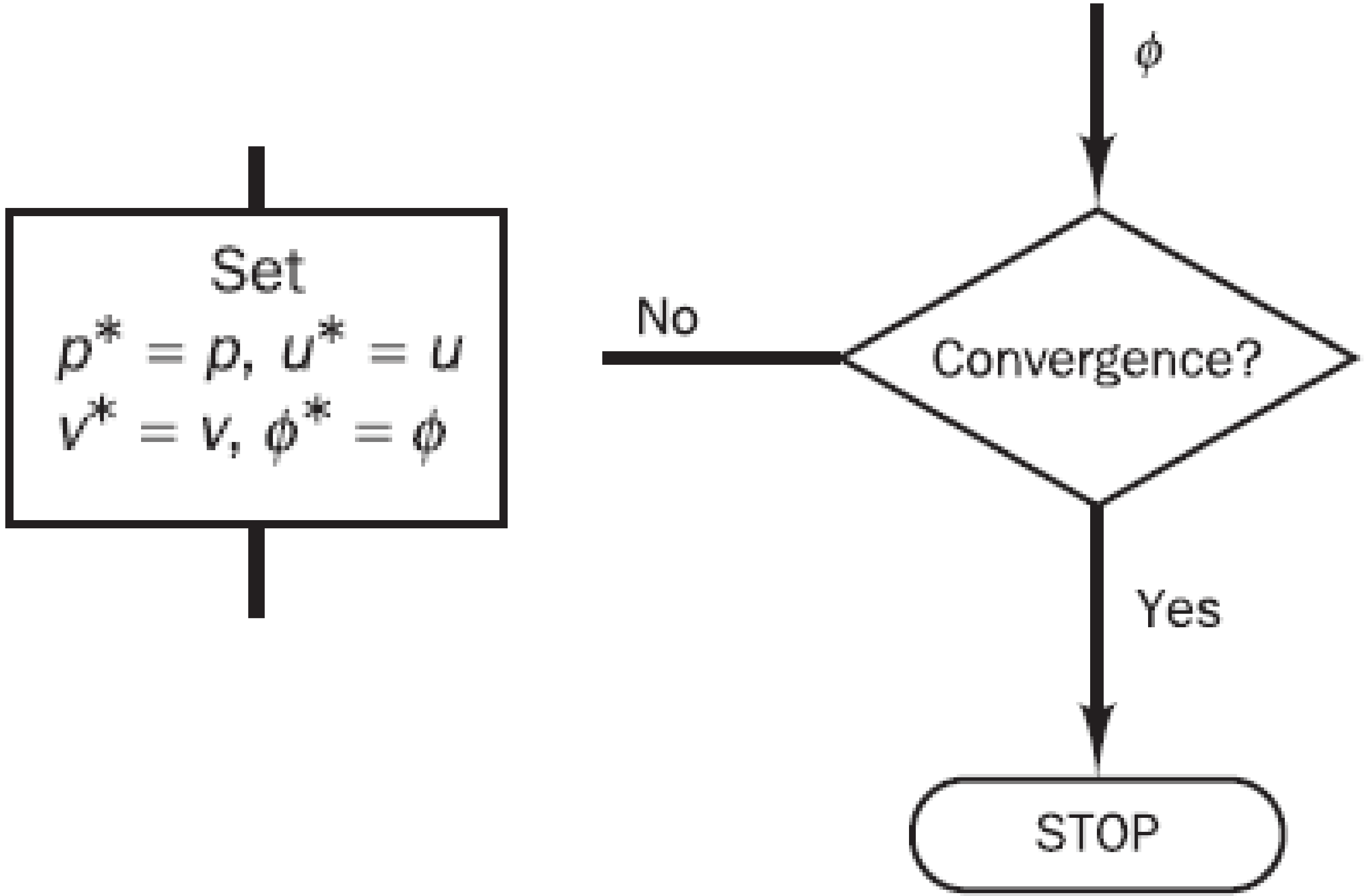
$p,\ u,\ v,\ \phi^*$

STEP 4: Solve all other discretised transport equations

$$a_{I,J}\phi_{I,J} = a_{I-1,J}\phi_{I-1,J} + a_{I+1,J}\phi_{I+1,J} + a_{I,J-1}\,\phi_{I,J-1} + a_{I,J+1}\,\phi_{I,J+1} + b_{\phi I,J}$$

ϕ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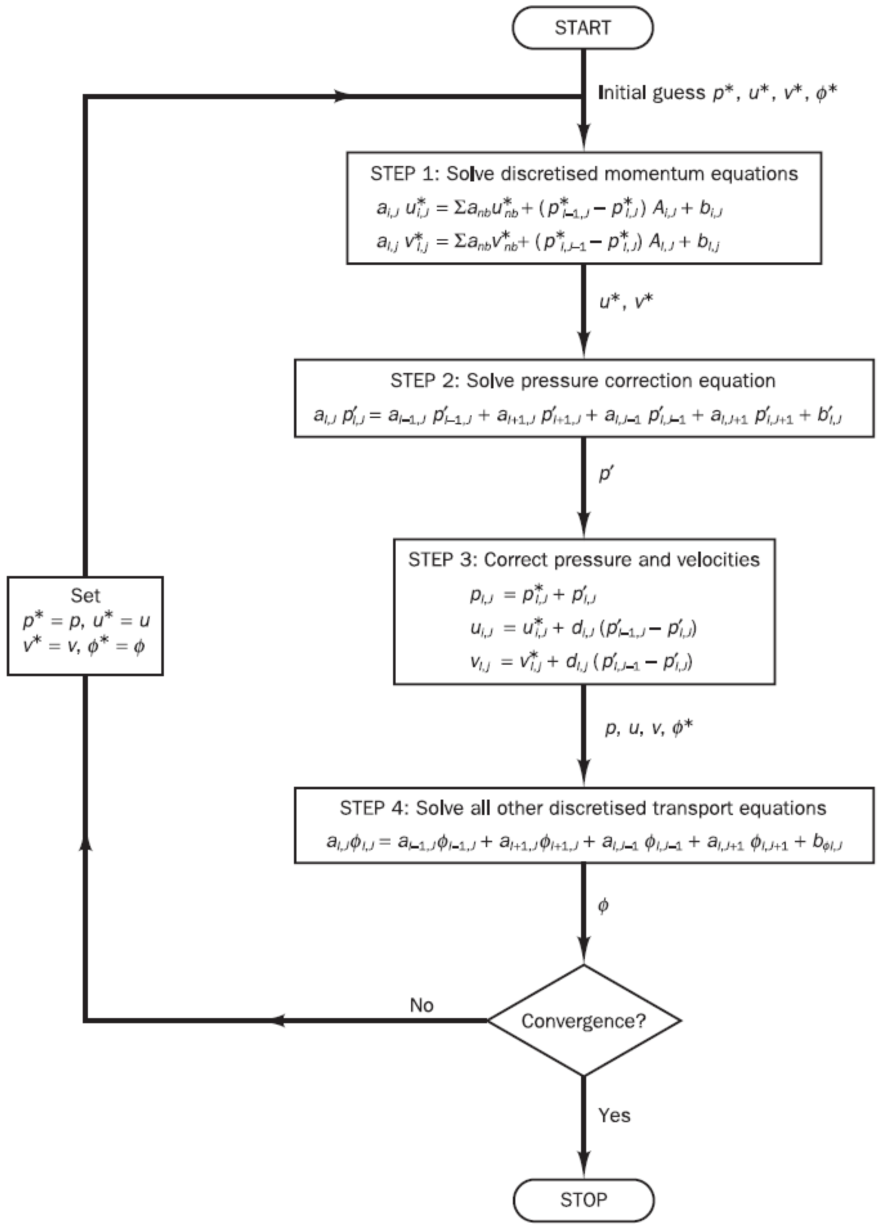
$v^* = v$, $\phi^* = \phi$
ϕ
No
Convergence?
Yes
STOP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Convergence?
No
Yes
STOP
Set
$p^* = p, u^* = u$
$v^* = v, \phi^* = \phi$

Other methods

SIMPLER
SIMPLEC — variation of SIMPLE
PISO

COUPLED - use Jacobeans of nonlinear velocity functions to form linear matrix (and avoid iteration)

Residual

Example:

$x-\exp(1/x)-2=0$

Find x using iteration

Explicit form 1:

$x=\exp(1/x)+2$

Solution process:

Guess x^0

Iteration :

$x^1=\exp(1/x^0)+2$, $R^1=x^1-x^0$
$X^2=\exp(1/x^1)+2$, $R^2=x^2-x^1$
……..
…….

Explicit form 2:

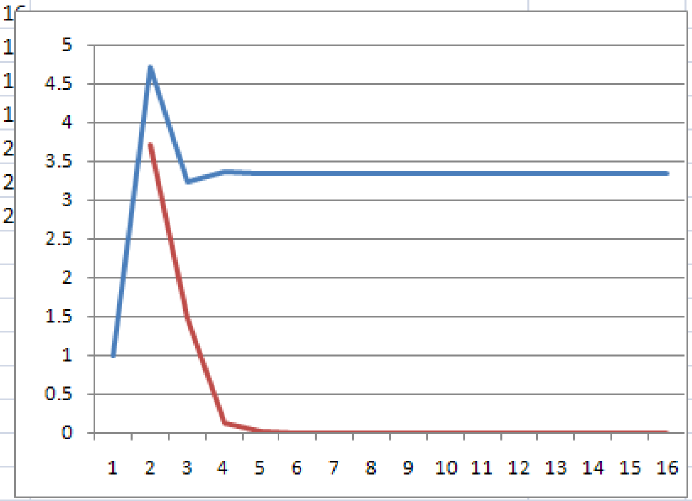
$x=1/(\ln(x)-\ln(2))$

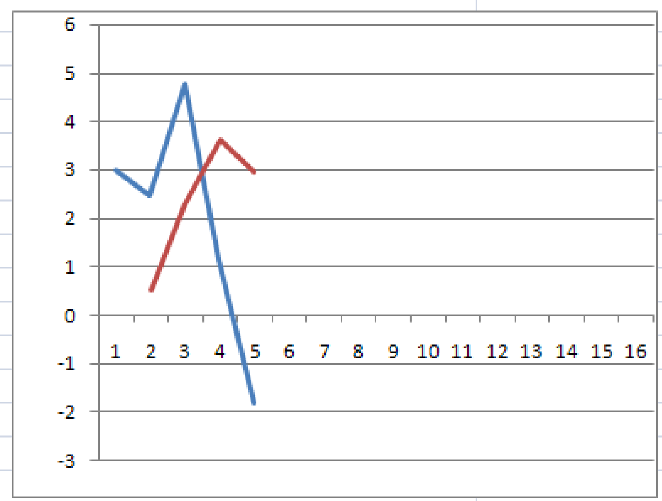
Not all iteration process converge!

See the example for the same equation

Convergence example

iteeration (n)	Function in explicit foramt x=exp(1/x)+2	residual	Function in explicit format x=1/(ln(x)-ln(2))	residual
n=0 , initial guess	1		3	
1	4.718281828	3.7182818	2.466303462	0.5336965
2	3.236075644	1.4822062	4.771600765	2.3052973
3	3.362084521	0.1260089	1.150040416	3.6215603
4	3.346400212	0.0156843	-1.807174174	2.9572146
5	3.348278472	0.0018783		
6	3.348052477	0.0002260		
7	3.348079654	0.0000272		
8	3.348076385	0.0000033		
9	3.348076778	0.0000004		
10	3.348076731	0.0000000		
11	3.348076737	0.0000000		
12	3.348076736	0.0000000		
13	3.348076736	0.0000000		
14	3.348076736	0.0000000		
15	3.348076736	0.0000000		





Residual calculation for CFD

- Residual for the cell

$$R_{\Phi_{ijk}} = \Phi^{k}_{ijk} - \Phi^{k-1}_{ijk}$$

iteration

cell position

Variable: p,V,T,…

- Total residual for the simulation domain

$$R_{\Phi total} = \Sigma |R_{\Phi_{ijk}}|$$

For all cells

- Scaled (normalized) residual

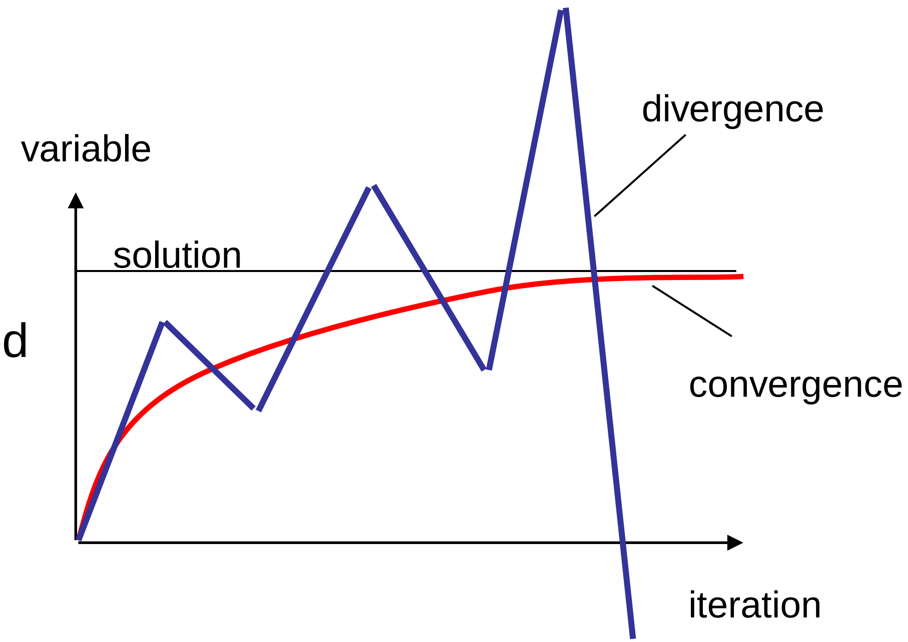
$$R_{\Phi} = \Sigma |R_{\Phi_{ijk}}| / F_{\Phi}$$

Flux of variable Φ used for normalization
Vary for different CFD software

Relaxation

Relaxation with iterative solvers:

When the equations are nonlinear
it can happen that you get divergency
in iterative procedure for solving considered
time step



Solution is Under-Relaxation:

$Y^{*}=f \cdot Y(n)+(1-f) \cdot Y(n-1)$ *Y – considered parameter , n –iteration , f – relaxation factor*

Value which is should be used for the next iteration

For our example $Y^{*}_{\text{in iteration } 101}=f \cdot Y(100)+(1-f) \cdot Y(99)$

f = [0-1] – under-relaxation -stabilize the iteration
f = [1-2] – over-relaxation - speed-up the convergence

Under-Relaxation is often required when you have nonlinear equations!

Example of relaxation

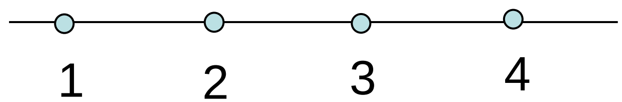
(example from homework 3 assignment)

Example: Advection diffusion equation, 1-D, steady-state, 4 nodes

$$a_N T_{N\text{-}1} + b_N T_N + c_N T_{N+1} = f_N$$

1) Explicit format:

$$T_N = 1/b_N f_N - a_N/b_N T_{N\text{-}1} - c_N/b_N T_{N+1}$$



2) Guess initial values:

$$T_1^0 = ..., \quad T_2^0 = .., \quad T_3^0 = .., \quad T_4^0 = ..$$

3) Substitute and calculate:

$$T^1{}_1 = 1/b_1 f_1 - c_1/b_1 T^0{}_2$$

$$T^1{}_2 = 1/b_2 f_2 - a_2/b_2 T^1{}_1 - c_2/b_2 T^0{}_3 \longrightarrow T_1^1 = ..., \quad T_2^1 = .., \quad T_3^1 = .., \quad T_4^1 = ..$$

$$T^1{}_3 = 1/b_3 f_3 - a_3/b_3 T^1{}_2 - c_3/b_3 T^0{}_4$$

$$T^1{}_4 = 1/b_4 f_4 - a_4/b_4 T^1{}_3$$

$$T_1^{1r} = fT_1^1 + (1\text{-}f)T_1^0, \quad T_2^{1r} = fT_2^1 + (1\text{-}f)T_2^0, \quad ....$$

4) Substitute and calculate: $\longrightarrow$ $T_1^2 = ..., \quad T_2^2 = .., \quad T_3^2 = .., \quad T_4^2 = ..$

$$T_1^{2r} = fT_1^2 + (1\text{-}f)T_1^1, \quad T_2^{2r} = fT_2^2 + (1\text{-}f)T_2^1, \quad ....$$

Substitute and calculate:

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